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A SUBCLASS OF ANALYTIC FUNCTIONS DEFINED BY MULTIPLIER TRANSFORMATION

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Abstract. In this paper, we consider the multiplier transformation

$$I_p(n,\lambda)f(z) = z^p + \sum_{k=p+1}^{\infty}\left(\frac{k+\lambda}{p+\lambda}\right)^n a_k z^k$$

where $p \in \mathbb{N}$, $n \in \mathbb{N} \cup 0, \lambda \geq 0$ and we provide the sufficient conditions for functions to be in the class $B(n,\mu,\alpha,\lambda)$.

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1. Introduction and Preliminaries

Let A_p denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \qquad p \in \mathbb{N} = \{1,2,...\}$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$.
Let S_p denote the subclass of functions that are univalent in U.
A function $f \in A_p$ is said to be p-valent starlike of order $\alpha\,(0 \leq \alpha < p)$ in U, if it satisfies the following inequality:

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \qquad z \in U.$$

We denote by $S_p^*(\alpha)$ the class of all such functions.
A function $f \in A_p$ is said to be p-valent convex of order $\alpha\,(0 \leq \alpha < p)$ in U, if and only if

$$\mathrm{Re}\left(\frac{zf''(z)}{f'(z)}+1\right) > \alpha, \qquad z \in U$$

for some $\alpha, (0 \leq \alpha < 1)$.
We denote by $K_p(\alpha)$ the class of all those functions $f \in A_p$ which are multivalently convex of order α in U and denote by $R(\alpha)$ the class of functions in A_p which satisfy

$$\mathrm{Re}f'(z) > \alpha, \qquad z \in U.$$

It is well known that $K_p(\alpha) \subset S_p^*(\alpha) \subset S_p$.
If f and g are analytic functions in U, we say that f is subordinate to g, written $f \prec g$ if $w(0) = 0, |w(z)| < 1$, for all $z \in U$. If g is univalent then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.
The following multiplier transformation was given by Sukhwinder Singh, Sushma Gupta and Sukhjit Singh [1].
Definition 1.([1]). *For $f \in A_p, p \in \mathbb{N}$, $n \in \mathbb{N} \cup 0, \lambda \geq 0$, the operator $I_p(n,\lambda)f(z)$ is defined by the following infinite series*

$$I_p(n,\lambda)f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{k+\lambda}{p+\lambda}\right)^n a_k z^k. \tag{1}$$

It is easily verified from (1) that

$$(p+\lambda)\, I_p(n+1,\lambda)f(z) = p\,(1-\lambda)\, I_p(n,\lambda)f(z) + \lambda z\,(I_p(n,\lambda)f(z))'. \tag{2}$$

Remark 1. *If $p = 1$ we have*

$$I_1(n,\lambda)f(z) = I(n,\lambda)$$

and

$$(\lambda+1)\, I(n+1,\lambda)f(z) = (1-\lambda)\, I(n,\lambda)f(z) + \lambda z\,(I(n,\lambda)f(z))',$$

for $z \in U$.
Remark 2. *If $f \in A_n$, $f(z) = z + \sum_{k=p+1}^{\infty} a_k z^k$, then*

$$I(n,\lambda)f(z) = z + \sum_{k=p+1}^{\infty} \left(\frac{k+\lambda}{p+\lambda}\right)^n a_k z^k,$$

for $z \in U$.
In the proof of our main result we need the following lemma.
Lemma 1. ([2]) . *Let u be analytic in U with $u(0) = 1$ and suppose that*

$$\mathrm{Re}\left(1 + \frac{zu'(z)}{u(z)}\right) > \frac{3\alpha - 1}{2\alpha}, z \in U. \tag{3}$$

Then $\mathrm{Re}u(z) > \alpha$ for $z \in U$ and $\frac{1}{2} \leq \alpha < 1$.

2. Main results

Definition 2. *We say that a function $f \in A_p$ is in the class $B(n,\mu,\alpha,\lambda)$, $n \in \mathrm{N}, \mu \geq 0, \alpha \in [0,1)$.*
If

$$\left|\frac{I(n+1,\lambda)}{z}\left(\frac{z}{I(n,\lambda)f(z)}\right)^{\mu} - 1\right| < 1 - \alpha, z \in U. \tag{4}$$

In this paper we provide a sufficient condition for functions to be in the class $B(n, \mu, \alpha, \lambda)$.

Theorem 1. *For the functions* $f \in A_p$, $n \in \mathbb{N}, \mu \geq 0, \frac{1}{2} \leq \alpha < 1$.
If

$$\frac{(\lambda+1)}{\lambda} \frac{I(n+2,\lambda)f(z)}{I(n+1,\lambda)f(z)} - \mu \frac{(\lambda+1)}{\lambda} \frac{I(n+1,\lambda)f(z)}{I(n,\lambda)f(z)} + \frac{1}{\lambda}(\mu-1) \prec 1+\beta z, z \in U \tag{5}$$

where

$$\beta = \frac{3\alpha - 1}{2\alpha}$$

then $f \in B(n, \mu, \alpha, \lambda)$.

Proof. If we consider

$$u(z) = \frac{I(n+1,\lambda)f(z)}{z} \left(\frac{z}{I(n,\lambda)f(z)} \right)^{\mu},$$

then $u(z)$ is analytic in U with $u(0) = 1$. A simple differentiation yields

$$\frac{zu'(z)}{u(z)} = \frac{(\lambda+1)}{\lambda} \frac{I(n+2,\lambda)f(z)}{I(n+1,\lambda)f(z)} - \frac{\mu(\lambda+1)}{\lambda} \frac{I(n+1,\lambda)f(z)}{I(n,\lambda)f(z)} + \frac{(\mu-1)}{\lambda}$$

Using (4) we get

$$\mathrm{Re}\left(1 + \frac{zu'(z)}{u(z)}\right) > \frac{3\alpha-1}{2\alpha}.$$

From Lemma 1. we have

$$\mathrm{Re}\left(\frac{I(n+1,\lambda)f(z)}{z} \left(\frac{z}{I(n,\lambda)f(z)} \right)^{\mu} \right) > \alpha.$$

Therefore, $f \in B(n, \mu, \alpha, \lambda)$, by Definition 2.

3. Applications of Theorem 1.

First of all, setting $n = 1, \mu = 1, \alpha = \frac{1}{2}, \lambda = 1$ in Theorem 1, we immediately arrive at the following application of Theorem 1. we have

Corollary 1. *If* $f \in A_1$ *and*

$$\mathrm{Re}\left(\frac{zf'(z) + 3z^2 f''(z) + z^3 f'''(z)}{zf'(z) + z^2 f''(z)} - \frac{zf'(z) + z^2 f''(z)}{zf'(z)} \right) > -\frac{1}{2}$$

then $f \in B\left(1, 1, \frac{1}{2}, 1\right)$.

Setting $n = 1, \mu = 0, \alpha = \frac{1}{2}, \lambda = 1$ we obtain the following interesting consequence of Theorem 1.

Corollary 2. *If* $f \in A_1$ *and*

$$\mathrm{Re}\left(\frac{zf'(z) + 3z^2 f''(z) + z^3 f'''(z)}{zf'(z) + z^2 f''(z)} \right) > -\frac{3}{2}$$

then $f \in B\left(1, 0, \frac{1}{2}, 1\right)$.

Setting $n = 0, \mu = 1, \alpha = \frac{1}{2}, \lambda = 1$ we obtain another consequence of Theorem 1.
Corollary 3. *If* $f \in A_1$ *and*

$$\mathrm{Re}\left(\frac{zf'(z) + z^2 f''(z)}{zf'(z)} - \frac{zf'(z)}{f(z)}\right) > -\frac{1}{2}$$

then $f \in B\left(0, 1, \frac{1}{2}, 1\right)$.

Finally, setting $n = 0, \mu = 0, \alpha = \frac{1}{2}, \lambda = 1$ we obtain the next consequence of Theorem 1.
Corollary 4. *If* $f \in A_1$ *and*

$$\mathrm{Re}\left(\frac{zf'(z) + z^2 f''(z)}{zf'(z)}\right) > \frac{3}{2}$$

then $f \in B\left(0, 0, \frac{1}{2}, 1\right)$.

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