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ON THE HYRES-ULAM-RASSIAS STABILITY OF A FUNCTIONAL EQUATION IN NON-ARCHIMEDEAN AND RANDOM NORMED SPACES

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Abstract. In this paper we prove the Hyres-Ulam-Rassias stability of the following functional equation

$$f(mx+ny)=\frac{(m+n)f(x+y)}{2}+\frac{(m-n)f(x-y)}{2} \tag{1}$$

where $m,n\in N$ with $m+n, m-n\neq 0$, in non-Archimedean and random normed spaces.
The concept of Hyers-Ulam-Rassias stability originated from Th. M. Rassias stability theorem that appeared in his paper: On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. 72 (1978), 297-300.

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1. Introduction

A classical question in the theory of functional equations is the following: When is it true that a function which approximately satisfies a functional equation D must be close to an exact solution of D?.
If the problem accepts a solution, we say that the equation D is stable. The first stability problem concerning group homomorphisms was raised by Ulam [32] in 1940. We are given a group G and a metric group G' with metric $d(.,.)$. Given $\varepsilon>0$, dose there exist a $\delta>0$ such that if $f:G\to G'$ satisfies $d(f(xy),f(x)f(y))<\delta$, for all $x,y\in G$, , then a homomorphism $h:G\to G$; exists with $d(f(x),h(x))<\varepsilon$ for all $x\in G$?.
In the next year D.H. Hyres [10], gave a positive answer to the above question for additive groups under the assumption that the groups are Banach spaces.
In 1978, Th. M. Rassias [25] proved a generalization of Hyres's theorem for additive mappings in the following way:

Theorem 1. Let f be an approximately additive mapping from a normed vector space E into a Banach space E', i.e., f satisfies the inequality

$$||f(x+y) - f(x) - f(y)|| \leq \varepsilon(||x||^r + ||y||^r) \tag{2}$$

for all $x, y \in E$, where ϵ, and r are constants with $\epsilon > 0$ and $0 \leq r < 1$. Then there exists a unique additive mapping $T : E \to F$ such that for all $x \in E$

$$||f(x) - T(x)|| \leq \frac{2\epsilon}{2 - 2^r}\|x\|^r \tag{3}$$

for all $x \in E$. The result of Th. M. Rassias has influenced the development of what is now called the Hyers-Ulam-Rassias stability theory for functional equations. In 1994, a generalization of Rassias' theorem was obtained by Găvruta [8] by replacing the bound $\epsilon(\|x\|^p + \|y\|^p)$ by a general control function $\phi(x, y)$.
Several stability results have been recently obtained for various equations, also for mappings with more general domains and ranges (see [1, 2, 4, 5, 7, 11, 12, 13, 14, 17, 18, 19, 20, 21, 24, 27]).
In 1897, Hensel [9] has introduced a normed space which does not have the Archimedean property. It turned out that non-Archimedean spaces have many nice applications [6, 15, 16, 21, 23].
In 2003, Radu[26] proved a generalization of theorem Hyres for Cauchy functional equation in random normed spaces and many authors proved stability of various functional equations in random normed space[3, 28].

2. Preliminaries

Definition 1. By a *non-Archimedean field* we mean a field K equipped with a function(valuation) $|.| : K \to [0, \infty)$ such that for all $r, s \in K$, the following conditions hold:
(i) $|r| = 0$ if and only if $r = 0$
(ii) $|rs| = |r||s|$
(iii) $|r + s| \leq max\{|r|, |s|\}$.
Definition 2. Let X be a vector space over a scalar field K with a non-Archimedean non-trivial valuation $|.|$. A function $||.|| : X \to R$ is a *non-Archimedean norm* (valuation) if it satisfies the following conditions:
(i) $||x|| = 0$ if and only if $x = 0$
(ii) $||rx|| = |r|||x||$ $(r \in K, x \in X)$
(iii) The strong triangle inequality(ultrametric); namely

$$||x + y|| \leq max\{||x||, ||y||\}. \quad x, y \in X$$

Then $(X, ||.||)$ is called a non-Archimedean space. Due to the fact that

$$||x_n - x_m|| \leq max\{||x_{j+1} - x_j|| : m \leq j \leq n-1\} \quad (n > m)$$

Definition 3. A sequence $\{x_n\}$ is *Cauchy* if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean space. By a complete non-Archimedean space we mean one in which every Cauchy sequence is convergent. The most important examples of non-Archimedean spaces are $p-$adic numbers. A key property of $p-$adic numbers is that they do not satisfy the Archimedean axiom: for all $x, y > 0$, there exists an integer n such that $x < ny$.

Example 1. Fix a prime number p. For any nonzero rational number x, there exists a unique integer $n_x \in Z$ such that $x = \frac{a}{b}p^{n_x}$, where a and b are integers not divisible by p. Then $|x|_p := p^{-n_x}$ defines a non-Archimedean norm on Q. The completion of Q with respect to the metric $d(x, y) = |x - y|_p$ is denoted by Q_p which is called the *p-adic number field.* In fact, Q_p is the set of all formal series $x = \sum_{k \geq n_x}^{\infty} a_k p^k$ where $|a_k| \leq p - 1$ are integers. The addition and multiplication between any two elements of Q_p are defined naturally. The norm $|\sum_{k \geq n_x}^{\infty} a_k p^k|_p = p^{-n_x}$ is a non-Archimedean norm on Q_p and it makes Q_p a locally compact filed.

Definition 4. A function $F : R \to [0, 1]$ is called a distribution function if it is nondecreasing and left-continuous, with $sup_{t \in R} F(t) = 1$ and $inf_{t \in R} F(t) = 0$. The class of all distribution functions F with $F(0) = 0$ is denoted by D_+.

Example 2. For every $a \geq 0$, H_a is the element of D_+ defined by

$$H_a(t) = \begin{cases} 0 & \text{if} \quad t \leq a \\ 1 & \text{if} \quad t > a \end{cases} . \tag{4}$$

Definition 5. Let X be a real vector space, Ψ be a mapping from X into D_+ (for any $x \in X$, $\Psi(x)$ is denoted by Ψ_x) and T be a t-norm. The triple (X, Ψ, T) is called a random normed space (briefly RN-space) iff the following conditions are satisfied:
(i) $\quad \Psi_x = H_0(t) \quad$ iff $\quad x = \theta$, the null vector;
(ii) $\Psi_{\alpha x}(t) = \Psi_x\left(\frac{t}{|\alpha|}\right)$ for all $\alpha \in R$, $\alpha \neq 0$ and $x \in X$.
(iii) $\Psi_{x+y}(t + s) \geq T(\Psi_x(t), \Psi_y(s))$, for all $x, y \in X$ and $t, s > 0$.

Every normed space $(X, ||.||)$ defines a random normed space (X, Ψ, T_M) where for every $t > 0$,

$$\Psi_u(t) = \frac{t}{t + ||u||}$$

and T_M is the minimum t-norm. This space is called the induced random normed space.
If the t-norm T is such that $sup_{0<a<1}T(a,a) = 1$, then every RN-space (X, Ψ, T) is a metrizable linear topological space with the topology τ (called the Ψ-topology or the (ϵ, δ)-topology) induced by the base of neighborhoods of θ, $\{U(\epsilon, \lambda)|\epsilon > 0, \lambda \in (0, 1)\}$, where

$$U(\epsilon, \lambda) = \{x \in X|\Psi_x(\epsilon) > 1 - \lambda\}$$

Definition 6. A sequence $\{x_n\}$ in an RN-space (X, Ψ, T) converges to $x \in X$, in the topology τ (we denote $lim x_n = x$) if $lim_{n\to\infty}\Psi_{x_n-x}(t) = 1$, $\forall t > 0$.
Definition 7. A sequence $\{x_n\}$ is called Cauchy sequence if for all $t > 0$,

$$lim_{n\to\infty}\Psi_{x_n-x_m}(t) = 1.$$

The RN-space (X, Ψ, T) is said to be complete if every Cauchy sequence in X is convergent.

3. Non-Archimedean Stability of Functional Equation (1)

Throughout this section, we prove the Hyers-Ulam-Rassias stability of the following functional equation

$$f(mx + ny) = \frac{(m + n)f(x + y)}{2} + \frac{(m - n)f(x - y)}{2}$$

where $m, n \in N$ with $m + n, m - n \neq 0$, in non-Archimedean normed space.
Throughout this section, Let H be an additive semigroup and X is a complete non-Archimedean normed space.

Theorem 2. Let $\psi : H^2 \to [0, +\infty)$ be a function such that

$$\lim_{p\to\infty} \frac{\psi(m^p x, m^p y)}{|m|^p} = 0; \quad x, y \in H, \tag{5}$$

and let for each $x \in H$ the limit

$$\Psi(x) = \lim_{p\to\infty} max\left\{\frac{\psi(m^k x, 0)}{|m|^k}\ ;\ 0 \leq k < p\right\} \tag{6}$$

exists. Suppose that $f : H \to X$ be a mapping satisfying

$$\left\|f(mx + ny) - \frac{(m + n)f(x + y)}{2} - \frac{(m - n)f(x - y)}{2}\right\|_X \leq \psi(x, y). \tag{7}$$

Then the limit

$$T(x) = \lim_{p\to\infty} \frac{f(m^p x)}{m^p}, \tag{8}$$

exists for all $x \in H$ and $T : H \to X$ is a mapping satisfying

$$\Big\| f(x) - T(x) \Big\|_X \le \frac{1}{|m|} \Psi(x). \quad x \in H \tag{9}$$

Moreover, if

$$\lim_{j\to\infty} \lim_{p\to\infty} max \left\{ \frac{\psi(m^k x, 0)}{|m|^k} \ ; \ j \le k < j+p \right\} = 0 \tag{10}$$

then T is the unique mapping satisfying (9).

Proof: Putting $y = 0$ in (7), we get

$$\left\| \frac{f(mx)}{m} - f(x) \right\|_X \le \frac{1}{|m|} \psi(x, 0). \tag{11}$$

Replacing x by $m^p x$ in (11) and dividing both sides by m^p, we get

$$\left\| \frac{f(m^{p+1}x)}{m^{p+1}} - \frac{f(m^p x)}{m^p} \right\|_X \le \frac{\psi(m^p x, 0)}{|m|^{p+1}} \tag{12}$$

for all $x \in H$. It follows from (5) and (12) that the sequence $\left\{ \frac{f(m^p x)}{m^p} \right\}_{p=1}^{+\infty}$ is a Cauchy sequence. Since X is complete, so the sequence $\left\{ \frac{f(m^p x)}{m^p} \right\}_{p=1}^{+\infty}$ is convergent. Set

$$T(x) := \lim_{p\to\infty} \frac{f(m^p x)}{m^p}.$$

Using induction we see that

$$\left\| \frac{f(m^p x)}{m^p} - f(x) \right\|_X \le \frac{1}{|m|} max \left\{ \frac{\psi(m^k x, 0)}{|m|^k} \ ; \ 0 \le k < p \right\}. \tag{13}$$

Indeed, (13) holds for $p = 1$ by (11). Let (13) holds for p, then we obtain

$$\begin{aligned} \left\| \frac{f(m^{p+1}x)}{m^{p+1}} - f(x) \right\|_X &= \left\| \frac{f(m^{p+1}x)}{m^{p+1}} \pm \frac{f(m^p x)}{m^p} - f(x) \right\|_X \\ &\le max \left\{ \left\| \frac{f(m^{p+1}x)}{m^{p+1}} - \frac{f(m^p x)}{m^p} \right\|_X, \left\| \frac{f(m^p x)}{m^p} - f(x) \right\|_X \right\} \end{aligned} \tag{14}$$

$$\begin{aligned} &\leq \frac{1}{|m|} max \left\{ \frac{\psi(m^p x, 0)}{|m|^p},\ max \left\{ \frac{\psi(m^k x, 0)}{|m|^k}\ ;\ 0 \leq k < p \right\} \right\} \\ &= \frac{1}{|m|} max \left\{ \frac{\psi(m^k x, 0)}{|m|^k}\ ;\ 0 \leq k < p+1 \right\}. \end{aligned}$$

So for all $p \in N$ and all $x \in H$, (13) holds. By taking p to approach infinity in (14) and using (6) one obtains (9). Replacing x by $m^p x$ and y by $m^p y$ respectively, in (7) and using (5), we obtain that

$$T(mx + ny) = \frac{(m+n)T(x+y)}{2} + \frac{(m-n)T(x-y)}{2}.$$

If S is another mapping satisfies (9), then for $x \in H$, we get

$$\begin{aligned} \Big\| T(x) - S(x) \Big\|_X &= \lim_{k\to\infty} \frac{1}{|m|^k} \Big\| T(m^k x) - S(m^k x) \Big\|_X \\ &\leq \lim_{k\to\infty} \frac{1}{|m|^k} max \left\{ \Big\| T(m^k x) - f(m^k x) \Big\|, \Big\| S(m^k x) - f(m^k x) \Big\|_X \right\} \\ &\leq \frac{1}{|m|} \lim_{j\to\infty} \lim_{p\to\infty} max \left\{ \frac{\psi(m^k x, 0)}{|m|^k}\ ;\ j \leq k < j+p \right\} = 0. \end{aligned}$$

Therefore $T = S$. This completes the proof of uniqueness of T.

Theorem 3. Let $\psi : H^2 \to [0, +\infty)$ be a function such that

$$\lim_{p\to\infty} |m|^p \psi\left(\frac{x}{m^p}, \frac{y}{m^p}\right) = 0; \qquad x, y \in H, \tag{15}$$

and let for each $x \in H$ the limit

$$\Theta(x) = \lim_{p\to\infty} max \left\{ |m|^{k+1} \psi\left(\frac{x}{m^{k+1}}, 0\right); 0 \leq k < p \right\}, \tag{16}$$

exists. Suppose that $f : H \to X$ is a mapping satisfying

$$\left\| f(mx+ny) - \frac{(m+n)f(x+y)}{2} - \frac{(m-n)f(x-y)}{2} \right\|_X \leq \psi(x, y). \tag{17}$$

Then the limit

$$T(x) = \lim_{p\to\infty} m^p f\left(\frac{x}{m^p}\right) \tag{18}$$

exists for all $x \in H$ and $T : H \to X$ is a mapping satisfying

$$\Big\| f(x) - T(x) \Big\|_X \leq \frac{1}{|m|} \Theta(x); \qquad x \in H \tag{19}$$

Moreover, if

$$\lim_{j\to\infty}\lim_{p\to\infty} max\left\{|m|^{k+1}\psi\left(\frac{x}{m^{k+1}},0\right); j\leq k<j+p\right\}=0, \tag{20}$$

then T is the unique mapping satisfying (19).

Proof: Letting $y=0$ in (17), we get

$$\Big\|f(mx)-mf(x)\Big\|_X\leq \psi(x,0), \tag{21}$$

for all $x\in H$. If we replace x by $\frac{x}{m^{p+1}}$ in (21), then we have

$$\left\|m^p f\left(\frac{x}{m^p}\right)-m^{p+1}f\left(\frac{x}{m^{p+1}}\right)\right\|_X\leq |m|^p\psi\left(\frac{x}{m^{p+1}},0\right), \tag{22}$$

for all $x\in H$ and all non-negative integer n. It follows from (15) and (22) that the sequence $\{m^p f\left(\frac{x}{m^p}\right)\}_{p=1}^{\infty}$ is a Cauchy sequence in X for all $x\in H$. Since X is complete, the sequence $\{m^p f\left(\frac{x}{m^p}\right)\}_{n=1}^{\infty}$ converges for all $x\in H$.
On the other hand, it follows from (22) that

$$\begin{aligned} &\left\|m^p f\left(\frac{x}{m^p}\right)-m^q f\left(\frac{x}{m^q}\right)\right\|_X \\ &=\left\|\sum_{k=p}^{q-1} m^{k+1}f\left(\frac{x}{m^{k+1}}\right)-m^k f\left(\frac{x}{m^k}\right)\right\|_X \\ &\leq max\left\{\left\|m^{k+1}f\left(\frac{x}{m^{k+1}}\right)-m^k f\left(\frac{x}{m^k}\right)\right\|_X \;;\; p\leq k<q-1\right\} \\ &\leq \frac{1}{|m|}max\left\{|m|^{k+1}\psi\left(\frac{x}{m^{k+1}},0\right); p\leq k<q-1\right\}, \end{aligned} \tag{23}$$

for all $x\in H$ and all non-negative integers p,q with $q>p\geq 0$. Letting $p=0$ and passing the limit $q\to\infty$ in the last inequality and using (16), we obtain (19).
The rest of the proof is similar to the proof of Theorem 2.

Corollary 1. Let $\gamma:[0,\infty)\to[0,\infty)$ be a function satisfying

$$\gamma\left(\frac{t}{|m|}\right)\leq\gamma\left(\frac{1}{|m|}\right)\gamma(t)\quad (t\geq 0),\quad \gamma\left(\frac{1}{|m|}\right)<\frac{1}{|m|}. \tag{24}$$

Let $\delta>0$ and $f:H\to X$ is a mapping satisfying

$$\left\|f(mx+ny)-\frac{(m+n)f(x+y)}{2}-\frac{(m-n)f(x-y)}{2}\right\|_X\leq\delta\left(\gamma(|x|)+\gamma(|y|)\right) \tag{25}$$

for all $x, y \in H$. Then there exists a unique mapping $T : H \to X$ such that

$$\Big\|f(x) - T(x)\Big\|_X \le \frac{\delta\gamma(|x|)}{|m|}; \quad x \in H, \tag{26}$$

Proof: Using induction one can show that for all $p \in N$,

$$\gamma\left(\frac{t}{|m|^p}\right) \le \gamma^p\left(\frac{1}{|m|}\right)\gamma(t) \le \frac{1}{|m|^p}\gamma(t). \tag{27}$$

Defining $\psi : H^2 \to [0, \infty)$ by $\psi(x, y) := \delta(\gamma(|x|) + \gamma(|y|))$. Since

$$|m|\gamma\left(\frac{1}{|m|}\right) < 1,$$

then we obtain that for all $x, y \in H$

$$\lim_{p\to\infty} |m|^p\psi\left(\frac{x}{m^p}, \frac{y}{m^p}\right) \le \lim_{n\to\infty}\left(|m|\gamma\left(\frac{1}{|m|}\right)\right)^p \psi(x, y) = 0.$$

Also

$$\Theta(x) = \lim_{p\to\infty} max\left\{|m|^{k+1}\psi\left(\frac{x}{m^{k+1}}, 0\right); 0 \le k < p\right\} = |m|\psi\left(\frac{x}{m}, 0\right), \tag{28}$$

and

$$\lim_{j\to\infty}\lim_{p\to\infty} max\left\{|m|^{k+1}\psi\left(\frac{x}{m^{k+1}}, 0\right); j \le k < j + p\right\} = \lim_{j\to\infty} |m|^{j+1}\psi\left(\frac{x}{m^{j+1}}, 0\right) = 0.$$

Hence the result follows by Theorem 3.

Example 3. Let $\delta > 0$, $0 < p < 1$ and $\gamma : [0, \infty) \to [0, \infty)$ defined by $\gamma(t) = t^p$. If $f : H \to X$ is a mapping satisfying

$$\left\|f(mx + ny) - \frac{(m + n)f(x + y)}{2} - \frac{(m - n)f(x - y)}{2}\right\|_X \le \delta(|x|^p + |y|^p); \;\; x, y \in H. \tag{29}$$

Then there exists a unique mapping $T : H \to X$ such that

$$\Big\|f(x) - T(x)\Big\|_X \le \frac{\delta|x|^p}{|m|}; \quad x \in H, \tag{30}$$

Corollary 2. Let $\gamma : [0, \infty) \to [0, \infty)$ is a function satisfying

$$\gamma(|m|t) \le \gamma(|m|)\gamma(t) \quad (t \ge 0), \quad \gamma(|m|) < |m| \tag{31}$$

Let $\delta > 0$ and $f : H \to X$ is a mapping satisfying

$$\Big\| f(mx+ny) - \frac{(m+n)f(x+y)}{2} - \frac{(m-n)f(x-y)}{2} \Big\|_X \leq \delta\Big(\gamma(|x|)+\gamma(|y|)\Big); \quad x, y \in H \tag{32}$$

Then there exists a unique mapping $T : H \to X$ such that

$$\Big\| f(x) - T(x) \Big\|_X \leq \frac{\delta\gamma(|x|)}{|m|}; \qquad x \in H, \tag{33}$$

Proof: Let $\psi : H^2 \to [0, \infty)$ be defined by $\psi(x, y) := \delta\Big(\gamma(|x| + \gamma(|y|)\Big)$.

4. Random Stability of Functional Equation (1)

Throughout this section, using direct method, we prove Hyers-Ulam-Rassias stability of functional equation (1) in random normed spaces.

Theorem 3. Let X be a vector space, (Z, Ψ, min) be an RN-space, and $\psi : X^2 \to Z$ be a function such that for some $0 < \alpha < m$,

$$\Psi_{\psi(mx,my)}(t) \geq \Psi_{\alpha\psi(x,y)}(t). \quad \forall x, y \in X, \ t > 0 \tag{34}$$

Also, for all $x, y \in X$ and $t > 0$

$$\lim_{n\to\infty} \Psi_{\psi(m^p x, m^p y)}(m^p t) = 1.$$

If (Y, μ, min) be a complete RN-space and $f : X \to Y$ is a mapping such that for all $x, y \in X$ and $t > 0$

$$\mu_{f(mx+ny) - \frac{(m+n)f(x+y)}{2} - \frac{(m-n)f(x-y)}{2}}(t) \geq \Psi_{\psi(x,y)}(t), \tag{35}$$

then there is a unique mapping $C : X \to Y$ such that

$$\mu_{f(x)-C(x)}(t) \geq \Psi_{\psi(x,0)}((m-\alpha)t). \tag{36}$$

Proof: Putting $y = 0$ in (35) we see that for all $x \in X$,

$$\mu_{\frac{f(mx)}{m} - f(x)}(t) \geq \Psi_{\psi(x,0)}(mt). \tag{37}$$

Replacing x by $m^p x$ in (37) and using (34), we obtain

$$\begin{aligned}\mu_{\frac{f(m^{p+1}x)}{m^{p+1}}-\frac{f(m^p x)}{m^p}}(t) &\geq \Psi_{\psi(m^p x,0)}(m^{p+1}t)\\ &\geq \Psi_{\psi(x,0)}\left(\frac{m^{p+1}t}{\alpha^p}\right). \end{aligned} \tag{38}$$

So by (38) we obtain

$$\begin{aligned}\mu_{\frac{f(m^p x)}{m^p}-f(x)}\left(\sum_{k=0}^{p-1}\frac{t\alpha^k}{m^{k+1}}\right) &= \mu_{\sum_{k=0}^{p-1}\frac{f(m^{k+1}x)}{m^{k+1}}-\frac{f(m^k x)}{m^k}}\left(\sum_{k=0}^{p-1}\frac{t\alpha^k}{m^{k+1}}\right)\\ &\geq T_{k=0}^{p-1}\left(\mu_{\frac{f(m^{k+1}x)}{m^{k+1}}-\frac{f(m^k x)}{m^k}}\left(\frac{t\alpha^k}{m^{k+1}}\right)\right)\\ &\geq T_{k=0}^{p-1}(\Psi_{\psi(x,0)}(t))\\ &= \Psi_{\psi(x,0)}(t).\end{aligned}$$

This implies that

$$\mu_{\frac{f(m^p x)}{m^p}-f(x)}(t) \geq \Psi_{\psi(x,0)}\left(\frac{t}{\sum_{k=0}^{p-1}\frac{\alpha^k}{m^{k+1}}}\right). \tag{39}$$

Replacing x by $m^q x$ in (39), we obtain

$$\begin{aligned}\mu_{\frac{f(m^{p+q}x)}{m^{p+q}}-\frac{f(m^q x)}{m^q}}(t) &\geq \Psi_{\psi(m^q x,0)}\left(\frac{t}{\sum_{k=0}^{p-1}\frac{\alpha^k}{m^{k+q+1}}}\right)\\ &\geq \Psi_{\psi(x,0)}\left(\frac{t}{\sum_{k=0}^{p-1}\frac{\alpha^{k+q}}{m^{k+q+1}}}\right)\\ &= \Psi_{\psi(x,0)}\left(\frac{t}{\sum_{k=q}^{p+q-1}\frac{\alpha^k}{m^{k+1}}}\right).\end{aligned} \tag{40}$$

As

$$\lim_{p,q\to\infty}\Psi_{\psi(x,0)}\left(\frac{t}{\sum_{k=q}^{p+q-1}\frac{\alpha^k}{m^{k+1}}}\right) = 1,$$

then $\left\{\frac{f(m^p x)}{m^p}\right\}_{n=1}^{+\infty}$ is a Cauchy sequence in complete RN-space (Y, μ, min), so there exist some point $C(x) \in Y$ such that $\lim_{n\to\infty}\frac{f(m^p x)}{m^p} = C(x)$. Fix $x \in X$ and put $q = 0$ in (40). Then we obtain

$$\mu_{\frac{f(m^p x)}{m^p}-f(x)}(t) \geq \Psi_{\psi(x,0)}\left(\frac{t}{\sum_{k=0}^{p-1}\frac{\alpha^k}{m^{k+1}}}\right). \tag{41}$$

and so, for every $\epsilon > 0$, we have

$$\begin{aligned}\mu_{C(x)-f(x)}(t+\epsilon) &\geq T\left(\mu_{C(x)-\frac{f(m^px)}{m^p}}(\epsilon), \mu_{\frac{f(m^px)}{m^p}-f(x)}(t)\right)\\ &\geq T\left(\mu_{C(x)-\frac{f(m^px)}{m^p}}(\epsilon), \Psi_{\psi(x,0)}\left(\frac{t}{\sum_{k=0}^{p-1}\frac{\alpha^k}{m^{k+1}}}\right)\right).\end{aligned}$$

Taking the limit as $p \to \infty$, we get

$$\mu_{C(x)-f(x)}(t+\epsilon) \geq \Psi_{\psi(x,0)}((m-\alpha)t). \tag{42}$$

Since ϵ was arbitrary by taking $\epsilon \to 0$ in (42), we obtain

$$\mu_{C(x)-f(x)}(t) \geq \Psi_{\psi(x,0)}((m-\alpha)t). \tag{43}$$

Replacing x and y by m^px and m^py respectively, in (35) and using this fact that $\lim_{p\to\infty}\Psi_{\psi(m^px,m^py)}(m^pt) = 1$, we get for all $x, y \in X$ and for all $t > 0$,

$$C(mx+ny) = \frac{(m+n)C(x+y)}{2} + \frac{(m-n)C(x-y)}{2}.$$

To prove the uniqueness of the mapping C, assume that there exist another mapping $D : X \to Y$ which satisfies (36). Since

$$\mu_{C(x)-D(x)}(t) = \lim_{n\to\infty}\mu_{\frac{C(m^px)}{m^p}-\frac{D(m^px)}{m^p}}(t). \tag{44}$$

So

$$\begin{aligned}\mu_{\frac{C(m^px)}{m^p}-\frac{D(m^px)}{m^p}}(t) &\geq min\left\{\mu_{\frac{C(m^px)}{m^p}-\frac{f(m^px)}{m^p}}\left(\frac{t}{2}\right), \mu_{\frac{D(m^px)}{m^p}-\frac{f(m^px)}{m^p}}\left(\frac{t}{2}\right)\right\}\\ &\geq \Psi_{\psi(m^px,0)}\left(\frac{m^p(m-\alpha)t}{2}\right)\\ &\geq \Psi_{\psi(x,0)}\left(\frac{m^p(m-\alpha)t}{2\alpha^p}\right).\end{aligned} \tag{45}$$

Since $\lim_{p\to\infty}\frac{m^p(m-\alpha)}{2\alpha^p} = \infty$, we get

$$\lim_{p\to\infty}\Psi_{\psi(x,0)}\frac{m^p(m-\alpha)t}{2\alpha^p} = 1.$$

Therefore, it follows that for all $t > 0$, $\mu_{C(x)-D(x)}(t) = 1$ and so $C(x) = D(x)$. This completes the proof.

Corollary 3. Let X be a real linear space, (Z, Ψ, min) be an RN-space and (Y, μ, min) a complete RN-space. Let $p \in (0, 1)$ and $z_0 \in Z$. If $f : X \to Y$ is a mapping such that for all $x, y \in X$ and $t > 0$

$$\mu_{f(mx+ny)-\frac{(m+n)f(x+y)}{2}-\frac{(m-n)f(x-y)}{2}}(t) \geq \Psi_{(||x||^p+||y||^p)z_0}(t), \tag{46}$$

then there is a unique mapping $C(x) : X \to Y$ such that

$$\mu_{f(x)-C(x)}(t) \geq \Psi_{||x||^p}\left(\frac{(m-m^p)t}{2}\right). \tag{47}$$

Proof: Let $\alpha = m^p$ and $\psi : X^2 \to Z$ be defined by $\psi(x, y) = (||x||^p + ||y||^p)z_0$.

Corollary 4. Let X be a real linear space, (Z, Ψ, min) be an RN-space and (Y, μ, min) a complete RN-space. Let $z_0 \in Z$. If $f : X \to Y$ is a mapping such that for all $x, y \in X$ and $t > 0$

$$\mu_{f(mx+ny)-\frac{(m+n)f(x+y)}{2}-\frac{(m-n)f(x-y)}{2}}(t) \geq \Psi_{\delta z_0}(t), \tag{48}$$

then there is a unique mapping $C : X \to Y$ such that for all $x \in X$ and $t > 0$

$$\mu_{f(x)-C(x)}(t) \geq \Psi_{\delta z_0}((m-1)t). \tag{49}$$

Proof: Let $\alpha = 1$ and $\psi : X^2 \to Z$ be defined by $\psi(x, y) = \delta z_0$.

References

[1] L. M. Arriola and W. A. Beyer, *Stability of the Cauchy functional equation over p-adic fields*, Real Anal. Exchange 31 (2005/06), no. 1, 125-132.

[2] H. Azadi Kenary, *Hyres-Rassias Stability of The Pexiderial Functional Equation* , to appear in Italian Journal of Pure and Applied Mathematics.

[3] H. Azadi Kenary, *The Probabilistic Stability of a Pexiderial Functional Equation in Random Normed Spaces,* to appear in Rendiconti Del Circolo Mathematico Di Palermo.

[4]Y. S. Cho and H. M. Kim, *Stability of functional inequalities with Cauchy-Jensen additive mappings*, Abstr. Appl. Anal. (2007), Art. ID 89180, 13 pp.

[5] S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific, River Edge, NJ, 2002.

[6] D. Deses, *On the representation of non-Archimedean objects*, Topology Appl. 153 (2005), no. 5-6, 774-785.

[7] W. Fechner, *Stability of a functional inequality associated with the Jordan-von Neumann functional equation*, Aequationes Math. 71 (2006), no. 1-2, 149-161.

[8] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. 184 (1994), no. 3, 431-436.

[9] K. Hensel, *Ubereine news Begrundung der Theorie der algebraischen Zahlen*, Jahresber. Deutsch. Math. Verein 6 (1897), 83-88.

[10] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A. 27 (1941), 222-224.

[11] D. H. Hyers, G. Isac, and Th. M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhauser, Basel, 1998.

[12] K. Jun and H. Kim, *On the Hyers-Ulam-Rassias stability problem for approximately k-additive mappings and functional inequalities*, Math. Inequal. Appl. 10 (2007), no. 4, 895-908.

[13] S.-M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, 2001.

[14]—- , *Hyers-Ulam-Rassias stability of Jensen's equation and its application*, Proc. Amer. Math. Soc. 126 (1998), no. 11, 3137-3143.

[15] A. K. Katsaras and A. Beoyiannis, *Tensor products of non-Archimedean weighted spaces of continuous functions*, Georgian Math. J. 6 (1999), no. 1, 33-44.

[16] A. Khrennikov, *Non-Archimedean Analysis: quantum paradoxes, dynamical systems and biological models*, Mathematics and its Applications, 427. Kluwer Academic Publishers, Dordrecht, 1997.

[17] Z. Kominek, *On a local stability of the Jensen functional equation*, Demonstratio Math. 22 (1989), no. 2, 499-507.

[18] L. Li, J. Chung, and D. Kim, *Stability of Jensen equations in the space of generalized functions*, J. Math. Anal. Appl. 299 (2004), no. 2, 578-586.

[19] A. K. Mirmostafaee, M. Mirzavaziri, and M. S. Moslehian, *Fuzzy stability of the Jensen functional equation*, Fuzzy Sets and Systems 159 (2008), no. 6, 730-738.

[20] A. K. Mirmostafaee and M. S. Moslehian, *Fuzzy versions of Hyers-Ulam-Rassias theorem*, Fuzzy Sets and Systems 159 (2008), no. 6, 720-729.

[21] Alireza Kamel Mirmostafaee , *Approximately Additive Mappings in Non-Archimedean Normed Spaces*, Bull. Korean Math. Soc. 46(2009), No.2, pp. 387-400.

[22] Abbas Najati and Asghar Rahimi, *Homomorphisms Between C^*-Algebras and Thier Stabilities*, Acta Universitatis Apulensis, N0 19/2009.

[23] P. J. Nyikos, *On some non-Archimedean spaces of Alexandrof and Urysohn*, Topology Appl. 91 (1999), 1-23.

[24] J. C. Parnami and H. L. Vasudeva, *On Jensen's functional equation*, Aequationes Math. 43 (1992), no. 2-3, 211-218.

[25] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. 72 (1978), no. 2, 297-300.

[26] V. Radu, *On the stability of the additive Cauchy functional functional equation in the random normed spaces*, J. Math. Anal. and Appl. 343(2008),567-572.

[27] J. Rätz, *On inequalities associated with the Jordan-von Neumann functional equation*, Aequationes Math. 66 (2003), no. 1-2, 191-200.

[28] R. Saadati, M. Vaezpour and Y.J. Cho, A *note to paper On the stability of cubic mappings and quartic mappings in random normed spaces*, J. of Inequalities and Applications,Vol. 2009, Article ID 214530.

[29] M. Sal Moslehian and T. M. Rassias, *Stability of functional equations in non- Archimedean spaces*, Appl. Anal. Discrete Math. 1 (2007), no. 2, 325-334.

[30] B. Schewizer and A. Sklar, *Probabilistic Metric Spaces*, North-Holland Series in Probability and Applied Mathematics, North-Holland, New York, NY, USA, 1983.

[31] F. Skof, *Local properties and approximation of operators*, Rend. Sem. Mat. Fis. Milano 53 (1983), 113-129.

[32] S. M. Ulam, *Problems in Modern Mathematics*, Science Editions, John Wiley and Sons, 1964.

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