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LIPSCHITZ CONTINUITY FOR MULTILINEAR COMMUTATOR OF LITTLEWOOD-PALEY OPERATOR

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Abstract. In this paper, we will study the continuity of multilinear commutator generated by Littlewood-paley operator and b on Triebel-Lizorkin space, Hardy space and Herz-Hardy space, where the function b belongs to Lipschiz space.

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1. Introduction

Let T be a Calderón-Zygmund operator, a well known result of Coifman, Rochberg and Weiss (see[4]) states that the commutator $[b,T](f)(x) = b(x)T(f)(x) - T(bf)(x)$ (where $b \in BMO$) is bounded on $L^p(R^n)$ for $1 < p < \infty$; Chanillo (see [2]) proves a similiar result when T is replaced by the fractional operators; In [7][14], Janson and Paluszynski study these result for the Triebel-Lizorkin spaces and the case $b \in Lip_\beta$, where Lip_β is the homogeneous Lipschitz space. The main purpose of this paper is to discuss the boundedness of multilinear commutator generated by Littlewood-paley operator and b on Triebel-Lizorkin space, Hardy space and Herz-Hardy space, where $b \in Lip_\beta$.

2. Preliminaries and Definitions

Throughout this paper, $M(f)$ will denote the Hardy-Littlewood maximal function of f, and write $M_p(f) = (M(f^p))^{1/p}$ for $0 < p < \infty$, Q will denote a cube of R^n with side parallel to the axes. Let $f_Q = |Q|^{-1}\int_Q f(x)dx$ and $f^\#(x) = \sup_{x\in Q}|Q|^{-1}\int_Q |f(y) - f_Q|dy$. Denote the Hardy spaces by $H^p(R^n)$. It is well known that $H^p(R^n)(0 < p \le 1)$ has the atomic decomposition characterization(see[15]).

For $\beta > 0$ and $p > 1$, let $\dot{F}_p^{\beta,\infty}(R^n)$ be the homogeneous Tribel-Lizorkin space. The Lipschitz space $Lip_\beta(R^n)$ is the space of functions f such that

$$||f||_{Lip_\beta} = \sup_{\substack{x,y\in R^n\\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\beta} < \infty.$$

Lemma 1.(see [14]) *For* $0<\beta<1,\ 1<p<\infty$, *we have*

$$\begin{aligned} ||f||_{\dot{F}_p^{\beta,\infty}} &\approx \left|\left|\sup_Q \frac{1}{|Q|^{1+\beta/n}}\int_Q |f(x)-f_Q|dx\right|\right|_{L^p} \\ &\approx \left|\left|\sup_{\cdot\in Q}\inf_c \frac{1}{|Q|^{1+\beta/n}}\int_Q |f(x)-c|dx\right|\right|_{L^p}. \end{aligned}$$

Lemma 2.(see [14]) *For* $0<\beta<1,\ 1\le p\le\infty$, *we have*

$$\begin{aligned} ||f||_{Lip_\beta} &\approx \sup_Q \frac{1}{|Q|^{1+\beta/n}}\int_Q |f(x)-f_Q|dx \\ &\approx \sup_Q \frac{1}{|Q|^{\beta/n}}\left(\frac{1}{|Q|}\int_Q |f(x)-f_Q|^p dx\right)^{1/p}. \end{aligned}$$

Lemma 3.(see [2]) *For* $1\le r<\infty$ *and* $\beta>0$, *let*

$$M_{\beta,r}(f)(x) = \sup_{x\in Q}\left(\frac{1}{|Q|^{1-\beta r/n}}\int_Q |f(y)|^r dy\right)^{1/r},$$

suppose that $r<p<\beta/n$, *and* $1/q = 1/p-\beta/n$, *then*

$$||M_{\beta,r}(f)||_{L^q} \le C||f||_{L^p}.$$

Lemma 4.(see [5]) *Let* $Q_1\subset Q_2$, *then*

$$|f_{Q_1}-f_{Q_2}| \le C||f||_{\dot{\Lambda}_\beta}|Q_2|^{\beta/n}.$$

Definition 1. *Let* $0<p,\ q<\infty$, $\alpha\in R$, $B_k=\{x\in R^n, |x|\le 2^k\}$, $A_k = B_k\backslash B_{k-1}$ *and* $\chi_k=\chi_{Ak}$ *for* $k\in\mathbf{Z}$, *where* χ_E *denote the characteristic function of the set* E.
1)The homogeneous Herz space is defined

$$\dot{K}_q^{\alpha,p}(R^n) = \{f\in L^q_{Loc}(R^n\backslash\{0\}) : ||f||_{\dot{K}_q^{\alpha,p}}<\infty\},$$

where

$$||f||_{\dot{K}_q^{\alpha,p}} = \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}||f\chi_k||_{L^q}^p\right]^{1/p};$$

2)The nonhomogeneous Herz space is defined by

$$K_q^{\alpha,p}(R^n)=\{f\in L^q_{Loc}(R^n),||f||_{K_q^{\alpha,p}}<\infty\},$$

where

$$||f||_{K_q^{\alpha,p}}=\left[\sum_{k=1}^{\infty}2^{k\alpha p}||f\chi_k||^p_{L^q}+||f\chi_{B_0}||^p_{L^q}\right]^{1/p}.$$

Definition 2. *Let $\alpha\in R$, $0<p,q<\infty$.*
(1) The homogeneous Herz type Hardy space is defined by

$$H\dot{K}_q^{\alpha,p}(R^n)=\{f\in S'(R^n):G(f)\in \dot{K}_q^{\alpha,p}(R^n)\},$$

and

$$||f||_{H\dot{K}_q^{\alpha,p}}=||G(f)||_{\dot{K}_q^{\alpha,p}};$$

(2) The nonhomogeneous Herz type Hardy space is defined by

$$HK_q^{\alpha,p}(R^n)=\{f\in S'(R^n):G(f)\in K_q^{\alpha,p}(R^n)\},$$

and

$$||f||_{HK_q^{\alpha,p}}=||G(f)||_{K_q^{\alpha,p}};$$

where $G(f)$ is the grand maximal function of f.
The Herz type Hardy spaces have the atomic decomposition characterization.
Definition 3. *Let $\alpha\in R$, $1<q<\infty$. A function $a(x)$ on R^n is called a central (α,q)-atom (or a central (a,q)-atom of restrict type), if*

1) *$Suppa\subset B(0,r)$ for some $r>0$ (or for some $r\geq 1$),*
2) *$||a||_{L^q}\leq |B(0,r)|^{-\alpha/n}$,*
3) *$\int_{R^n}a(x)x^\eta dx=0$ for $|\eta|\leq[\alpha-n(1-1/q)]$.*

Lemma 5.(see [6][13]) *Let $0<p<\infty$, $1<q<\infty$ and $\alpha\geq n(1-1/q)$. A temperate distribution f belongs to $H\dot{K}_q^{\alpha,p}(R^n)$(or $HK_q^{\alpha,p}(R^n)$) if and only if there exist central (α,q)-atoms(or central (α,q)-atoms of restrict type)a_j supported on $B_j=B(0,2^j)$ and constants λ_j, $\sum_j|\lambda_j|^p<\infty$ such that $f=\sum_{j=-\infty}^{\infty}\lambda_ja_j$(or $f=\sum_{j=0}^{\infty}\lambda_ja_j$)in the $S'(R^n)$ sense, and*

$$||f||_{H\dot{K}_q^{\alpha,p}}(or\ ||f||_{HK_q^{\alpha,p}})\sim\left(\sum_j|\lambda_j|^p\right)^{1/p}.$$

Definition 4. *Let $n>\delta>0$, $\varepsilon>0$ and ψ be a fixed function which satisfies the following properties:*
1) $\int_{R^n}\psi(x)dx=0$,

2) $|\psi(x)| \le C(1+|x|)^{-(n+1-\delta)}$,
3) $|\psi(x+y)-\psi(x)| \le C|y|^{\varepsilon}(1+|x|)^{-(n+1+\varepsilon-\delta)}$ *when* $2|y|<|x|$.
Let m be a positive integer and $b_j (1 \le j \le m)$ be the locally integrable function, set $\vec{b}=(b_1,\cdots,b_m)$. The multilinear commutator of Littlewood-Paley operator is defined by

$$S_\delta^{\vec{b}}(f)(x)=\left(\int\int_{\Gamma(x)}|F_t^{\vec{b}}(x,y)|^2\frac{dydt}{t^{n+1}}\right)^{1/2},$$

where $\Gamma(x)=\{(y,t)\in R_+^{n+1}: |x-y|<t\}$,

$$F_t^{\vec{b}}(f)(x,y)=\int_{R^n}\left[\prod_{j=1}^m(b_j(x)-b_j(z))\right]\psi_t(y-z)f(z)dz,$$

and $\psi_t(x)=t^{-n+\delta}\psi(x/t)$ for $t>0$. Set $F_t(f)=\psi_t * f$. We also define that

$$S_\delta(f)(x)=\left(\int\int_{\Gamma(x)}|F_t(f)(y)|^2\frac{dydt}{t^{n+1}}\right)^{1/2},$$

which is the Littlewood-Paley function.
Let H be the space $H=\{h: ||h||=(\int\int_{R_+^{n+1}}|h(y,t)|^2dydt/t^{n+1})^{1/2}<\infty\}$, then, for each fixed $x\in R^n$ $F_t(f)(x)$ may be viewed as a mapping from $[0,+\infty)$ to H, and it is clear that

$$S_\delta(f)(x)=||\chi_{\Gamma(x)}F_t(f)(y)|| \text{ and } S_\delta^{\vec{b}}(f)(x)=||\chi_{\Gamma(x)}F_t^{\vec{b}}(f)(x,y)||.$$

Note that when $b_1=\cdots=b_m$, $S_\delta^{\vec{b}}$ is just the m order commutator. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors(see [1-4][7-11][14]). Our main purpose is to establish the boundedness of the multilinear commutator on Triebel-Lizorkin space, Hardy space and Herz-Hardy space.
Given a positive integer m and $1\le j\le m$, we set $||\vec{b}||_{Lip_\beta}=\prod_{j=1}^m||b_j||_{Lip_\beta}$ and denote by C_j^m the family of all finite subsets $\sigma=\{\sigma(1),\cdots,\sigma(j)\}$ of $\{1,\cdots,m\}$ of j different elements. For $\sigma\in C_j^m$, set $\sigma^c=\{1,\cdots,m\}\setminus\sigma$. For $\vec{b}=(b_1,\cdots,b_m)$ and $\sigma=\{\sigma(1),\cdots,\sigma(j)\}\in C_j^m$, set $\vec{b}_\sigma=(b_{\sigma(1)},\cdots,b_{\sigma(j)})$, $b_\sigma=b_{\sigma(1)}\cdots b_{\sigma(j)}$ and $||\vec{b}_\sigma||_{Lip_\beta}=||b_{\sigma(1)}||_{Lip_\beta}\cdots||b_{\sigma(j)}||_{Lip_\beta}$.

2. Theorems and Proofs

Theorem 1. *Let* $0 < \beta < 1/2m$, $1 < p < n/\delta$, $1/p - 1/q = \delta/n$ *and* $\vec{b} = (b_1, \cdots, b_m)$ *with* $b_j \in Lip_\beta(R^n)$ *for* $1 \le j \le m$. *Then* $S_\delta^{\vec{b}}$ *is bounded from* $L^p(R^n)$ *to* $\dot{F}_q^{m\beta,\infty}(R^n)$.

Theorem 2. *Let* $0 < \delta < n - m\beta$, $0 < \beta < 1/2m$, $1 < p < n/(\delta + m\beta)$, $1/p - 1/q = (\delta + m\beta)/n$ *and* $\vec{b} = (b_1, \cdots, b_m)$ *with* $b_j \in Lip_\beta(R^n)$ *for* $1 \le j \le m$. *Then* $S_\delta^{\vec{b}}$ *is bounded from* $L^p(R^n)$ *to* $L^q(R^n)$.

Theorem 3. *Let* $0 < \delta < n - m\beta$, $0 < \beta \le 1$, $\max(n/(n+m\beta), n/(n+m\varepsilon)) < p \le 1$, $1/p - 1/q = (\delta + m\beta)/n$, $\vec{b} = (b_1, \cdots, b_m)$ *with* $b_j \in Lip_\beta(R^n)$ *for* $1 \le j \le m$. *Then* $S_\delta^{\vec{b}}$ *is bounded from* $H^p(R^n)$ *to* $L^q(R^n)$.

Theorem 4. *Let* $0 < \delta < n - m\beta$, $0 < \beta \le 1$, $0 < p < \infty$, $1 < q_1, q_2 < \infty$, $1/q_1 - 1/q_2 = (\delta + m\beta)/n$, $n(1 - 1/q_1) \le \alpha < n(1 - 1/q_1) + m\beta$, $\vec{b} = (b_1, \cdots, b_m)$ *with* $b_j \in Lip_\beta(R^n)$ *for* $1 \le j \le m$. *Then* $S_\delta^{\vec{b}}$ *is bounded from* $H\dot{K}_{q_1}^{\alpha,p}(R^n)$ *to* $\dot{K}_{q_2}^{\alpha,p}(R^n)$.

Proof of Theorem 1. Fixed a cube $Q = (x_0, l)$ and $x \in Q$. Set $\vec{b}_Q = ((b_1)_Q, \cdots, (b_m)_Q)$, where $(b_j)_Q = |Q|^{-1} \int_Q b_j(y)dy$, $1 \le j \le m$. Write $f = f_1 + f_2$, where $f_1 = f\chi_{2Q}, f_2 = f\chi_{R^n \backslash 2Q}$, we have

$$\begin{aligned}
F_t^{\vec{b}}(f)(x,y) &= \int_{R^n} \left[\prod_{j=1}^m (b_j(x) - b_j(z))\right] \psi_t(y-z)f(z)dz \\
&= \int_{R^n} \prod_{j=1}^m [(b_j(x) - (b_j)_Q) - (b_j(z) - (b_j)_Q)]\psi_t(y-z)f(z)dz \\
&= \sum_{j=o}^m \sum_{\sigma \in C_j^m} (-1)^{m-j}(b(x) - b_Q)_\sigma \int_{R^n} (b_j(z) - b_Q)_{\sigma^c}\psi_t(y-z)f(z)dz \\
&= (b_1(x) - (b_1)_Q)\cdots(b_m(x) - (b_m)_Q)F_t(f)(y) \\
&\quad +(-1)^m F_t((b_1 - (b_1)_Q)\cdots(b_m - (b_m)_Q)f)(y) \\
&\quad +\sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} (-1)^{m-j}(b(x) - b_Q)_\sigma \int_{R^n} (b(z) - b_Q)_{\sigma^c}\psi_t(y-z)f(z)dz \\
&= (b_1(x) - (b_1)_Q)\cdots(b_m(x) - (b_m)_Q)F_t(f)(y) \\
&\quad +(-1)^m F_t((b_1 - (b_1)_Q)\cdots(b_m - (b_m)_Q)f_1)(y) \\
&\quad +(-1)^m F_t((b_1 - (b_1)_Q)\cdots(b_m - (b_m)_Q)f_2)(y) \\
&\quad +\sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} (-1)^{m-j}(b(x) - b_Q)_\sigma F_t((b - b_Q)_{\sigma^c}f)(x,y),
\end{aligned}$$

thus

$$\begin{aligned}
& |S_\delta^{\vec{b}}(f)(x) - S_\delta(((b_1)_Q - b_1)\cdots((b_m)_Q - b_m)f_2)(x_0)| \\
\le\ & ||\chi_{\Gamma(x)}F_t^{\vec{b}}(f)(x,y) - \chi_{\Gamma(x_0)}F_t(((b_1)_Q - b_1)\cdots((b_m)_Q - b_m)f_2)(y)|| \\
\le\ & ||\chi_{\Gamma(x)}(b_1(x) - (b_1)_Q)\cdots(b_m(x) - (b_m)_Q)F_t(f)(y)|| \\
& + \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m} ||\chi_{\Gamma(x)}(b(x) - \vec{b}_Q)_\sigma F_t((b - b_Q)_{\sigma^c} f)(x,y)|| \\
& + ||\chi_{\Gamma(x)}F_t((b_1 - (b_1)_Q)\cdots(b_m - (b_m)_Q)f_1)(y)|| \\
& + ||\chi_{\Gamma(x)}F_t(\prod_{j=1}^m (b_j - (b_j)_Q)f_2)(y) - \chi_{\Gamma(x_0)}F_t(\prod_{j=1}^m (b_j - (b_j)_Q)f_2)(y)|| \\
=\ & I_1(x) + I_2(x) + I_3(x) + I_4(x),
\end{aligned}$$

so

$$\begin{aligned}
& \frac{1}{|Q|^{1+m\beta/n}}\int_Q |S_\delta^{\vec{b}}(f)(x) - S_\delta(((b_1)_Q - b_1)\cdots((b_m)_Q - b_m)f_2)(x_0)|dx \\
\le\ & \frac{1}{|Q|^{1+m\beta/n}}\int_Q I_1(x)dx + \frac{1}{|Q|^{1+m\beta/n}}\int_Q I_2(x)dx \\
& + \frac{1}{|Q|^{1+m\beta/n}}\int_Q I_3(x)dx + \frac{1}{|Q|^{1+m\beta/n}}\int_Q I_4(x)dx \\
=\ & I + II + III + IV.
\end{aligned}$$

For I, by using Lemma 2, we have

$$\begin{aligned}
I =\ & \frac{1}{|Q|^{1+m\beta/n}}\int_Q ||\chi_{\Gamma(x)}(b_1(x) - (b_1)_Q)\cdots(b_m(x) - (b_m)_Q)F_t(y)||dx \\
=\ & \frac{1}{|Q|^{1+m\beta/n}}\int_Q \left(\int\int_{R_+^{n+1}} |\chi_{\Gamma(x)}(b_1(x) - (b_1)_Q)\cdots(b_m(x) - (b_m)_Q F_t(y))|^2 \frac{dydt}{t^{n+1}}\right)^{1/2} dx \\
=\ & \frac{1}{|Q|^{1+m\beta/n}}\int_Q |(b_1(x) - (b_1)_Q)\cdots(b_m(x) - (b_m)_Q)||S_\delta(f)(x)|dx \\
\le\ & \frac{1}{|Q|^{1+m\beta/n}}\sup_{x\in Q}|b_1(x) - (b_1)_Q|\cdots|b_m(x) - (b_m)_Q|\int_Q |S_\delta(f)(x)|dx \\
\le\ & C||\vec{b}||_{Lip_\beta}\frac{1}{|Q|^{1+m\beta/n}}|Q|^{m\beta/n}\int_Q |S_\delta(f)(x)|dx \\
\le\ & C||\vec{b}||_{Lip_\beta}M(S_\delta(f))(x).
\end{aligned}$$

Fixed $1 < r < p$ and s with $1 < r < n/\delta$ and $1/s = 1/r - \delta/n$. For II, let μ, μ' be the integers such that $\mu + \mu' = m,\ 0 \le \mu < m,\ 0 < \mu' \le m$. By using the Hölder's inequality, **Lemma 2** and the (L^r, L^s) -boundedness of S_δ, we get

$$\begin{aligned}
II &= \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|Q|^{1+m\beta/n}}\int_Q ||\chi_{\Gamma(x)}(\vec{b}(x)-\vec{b}_Q)_\sigma F_t((\vec{b}-\vec{b}_Q)_{\sigma^c}f)(x,y)||dx \\
&\le \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|Q|^{1+m\beta/n}}\int_Q |(\vec{b}(x)-\vec{b}_Q)_\sigma||S_\delta((\vec{b}-\vec{b}_Q)_\sigma^c f)(x)|dx \\
&\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|Q|^{1+m\beta/n}}\left(\int_Q |\vec{b}(x)-\vec{b}_Q|_\sigma^{s'}dx\right)^{1/s'}\left(\int_{R^n}|S_\delta((\vec{b}-\vec{b}_Q)_{\sigma^c}f)(x)|^s\chi_Q dx\right)^{1/s} \\
&\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|Q|^{1+m\beta/n}}\left(\int_Q |\vec{b}(x)-\vec{b}_Q|_\sigma^{s'}dx\right)^{1/s'}\left(\int_{R^n}|(b(\vec{x})-\vec{b}_Q)_{\sigma^c}f(x)|^r\chi_Q dx\right)^{1/r} \\
&\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|Q|^{1+m\beta/n}}||\vec{b}_\sigma||_{Lip_\beta}|Q|^{\mu\beta/n}|Q|^{\mu\beta/n}||\vec{b}_{\sigma^c}||_{Lip_\beta}|Q|^{\mu'\beta/n}\left(\int_Q|f(x)|^r dx\right)^{1/r} \\
&= C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||\vec{b}_\sigma||_{Lip_\beta}||\vec{b}_{\sigma^c}||_{Lip_\beta}\left(\frac{1}{|Q|^{1-r\delta/n}}\int_Q|f(x)|^r dx\right)^{1/r} \\
&\le C||\vec{b}||_{Lip_\beta}M_{\delta,r}(f)(x).
\end{aligned}$$

For III, by Hölder's inequality, we have

$$\begin{aligned}
III &= \frac{1}{|Q|^{1+m\beta/n}}\int_Q|\chi_{\Gamma(x)}F_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_1)(y)|dx \\
&= \frac{1}{|Q|^{1+m\beta/n}}\int_Q|S_\delta((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_1)(x)|dx \\
&\le C\frac{1}{|Q|^{1+m\beta/n}}\left(\int_{R^n}|S_\delta(\prod_{j=1}^m(b_j-(b_j)_Q)f_1)(x)|^s dx\right)^{1/s}|Q|^{1-1/s} \\
&\le C\frac{1}{|Q|^{1+m\beta/n}}|Q|^{1-1/s}\left(\int_{2Q}|\prod_{j=1}^m(b_j(x)-(b_j)_Q)f(x)|^r dx\right)^{1/r} \\
&\le C\frac{1}{|Q|^{1+m\beta/n}}|Q|^{1-1/s}||\vec{b}||_{Lip_\beta}|Q|^{m\beta/n}\left(\int_{2Q}|f(x)|^r dx\right)^{1/r} \\
&\le C||\vec{b}||_{Lip_\beta}M_{\delta,r}(f)(x).
\end{aligned}$$

For IV, since $|x_0-y|\approx|x-y|$ for $y\in(2Q)^c$, by **Lemma 4** and the condition of ψ, we have

$$\begin{aligned}
I_4(x) &= ||\chi_{\Gamma(x)}F_t(\prod_{j=1}^m(b_j-(b_j)_Q)f_2)(y)-\chi_{\Gamma(x_0)}F_t(\prod_{j=1}^m(b_j-(b_j)_Q)f_2)(y)|| \\
&\le \left[\int\int_{R_+^{n+1}}(\int_{(2Q)^c}|\chi_{\Gamma(x)}-\chi_{\Gamma(x_0)}||f(z)|\prod_{j=1}^m|b_j(z)-(b_j)_Q|dz)^2\frac{dydt}{t^{n+1}}\right]^{1/2} \\
&\le C\int_{(2Q)^c}|f(z)|\prod_{j=1}^m|b_j(z)-(b_j)_Q| \\
&\times \left|\int\int_{|x-y|\le t}\frac{t^{1-n}dydt}{(t+|y-z|)^{2n+2-2\delta}}-\int\int_{|x_0-y|\le t}\frac{t^{1-n}dydt}{(t+|y-z|)^{2n+2-2\delta}}\right|^{1/2}dz \\
&\le C\int_{(2Q)^c}|f(z)|\prod_{j=1}^m|b_j(z)-(b_j)_Q| \\
&\times \left(\int\int_{|y|\le t,|x+y-z|\le t}\left|\frac{1}{(t+|x+y-z|)^{2n+2-2\delta}}-\frac{1}{(t+|x_0+y-z|)^{2n+2-2\delta}}\right|\frac{dydt}{t^{n-1}}\right)^{1/2}dz \\
&\le C\int_{(2Q)^c}|f(z)|\prod_{j=1}^m|b_j(z)-(b_j)_Q| \\
&\times \left(\int\int_{|y|\le t,|x+y-z|\le t}\frac{|x-x_0|t^{1-n}}{(t+|x+y-z|)^{2n+3-2\delta}}dydt\right)^{1/2}dz,
\end{aligned}$$

note that $2t+|x+y-z|\ge 2t+|x-z|-|y|\ge t+|x-z|$ when $|y|\le t$ and

$$\int_0^\infty\frac{tdt}{(t+|x-z|)^{2n+3-2\delta}}=C|x-z|^{-2n-1+2\delta},$$

then for $x\in Q$,

$$\begin{aligned}
I_4(x) &\le C\int_{(2Q)^c}|f(z)|\prod_{j=1}^m|b_j(z)-(b_j)_Q|\left(\int\int_{|y|\le t}\frac{|x_0-x|t^{1-n}2^{2n+3-2\delta}dydt}{(2t+2|x+y-z|)^{2n+3-2\delta}}\right)^{1/2}dz \\
&\le C\int_{(2Q)^c}|f(z)|\prod_{j=1}^m|b_j(z)-(b_j)_Q||x-x_0|^{1/2}\left(\int\int_{|y|\le t}\frac{t^{1-n}dydt}{(2t+|x+y-z|)^{2n+3-2\delta}}\right)^{1/2}dz \\
&\le C\int_{(2Q)^c}|f(z)|\prod_{j=1}^m|b_j(z)-(b_j)_Q||x-x_0|^{1/2}\left(\int\int_{|y|\le t}\frac{t^{1-n}dydt}{(t+|x-z|)^{2n+3-2\delta}}\right)^{1/2}dz \\
&\le C\int_{(2Q)^c}|f(z)|\prod_{j=1}^m|b_j(z)-(b_j)_Q||x-x_0|^{1/2}\left(\int_0^\infty\frac{tdt}{(t+|x-z|)^{2n+3-2\delta}}\right)^{1/2}dz
\end{aligned}$$

$$
\begin{aligned}
&\leq C\int_{(2Q)^c}|f(z)|\prod_{j=1}^{m}|b_j(z)-(b_j)_Q|\frac{|x-x_0|^{1/2}}{|x_0-z|^{n+1/2-\delta}}dz\\
&\leq C\sum_{k=1}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}|x_0-x|^{1/2}|x_0-z|^{-(n+1/2-\delta)}|f(z)|\prod_{j=1}^{m}|b_j(z)-(b_j)_Q|dz\\
&\leq C\sum_{k=1}^{\infty}2^{-k/2}|2^{k+1}Q|^{-1}\int_{2^{k+1}Q}|f(z)|\prod_{j=1}^{m}(|b_j(z)-(b_j)_{2^{k+1}Q}|+|(b_j)_{2^{k+1}Q}-(b_j)_Q|)dz\\
&\leq C\sum_{k=1}^{\infty}2^{-k/2}|2^{k+1}Q|^{m\beta/n}||\vec{b}||_{Lip_\beta}M_{\delta,r}(f)(x)\\
&\leq C||\vec{b}||_{Lip_\beta}|Q|^{m\beta/n}M_{\delta,r}(f)(x)\sum_{k=1}^{\infty}2^{(m\beta-1/2)k}\\
&\leq C||\vec{b}||_{Lip_\beta}|Q|^{m\beta/n}M_{\delta,r}(f)(x),
\end{aligned}
$$

so

$$IV\leq C||\vec{b}||_{Lip_\beta}M_{\delta,r}(f)(x).$$

We put these estimates together, by using **Lemma 1** and taking the supremum over all Q such that $x\in Q$, we obtain

$$||S_\delta^{\vec{b}}(f)(x)||_{\dot{F}_q^{m\beta,\infty}}\leq C||\vec{b}||_{Lip_\beta}||f||_{L^p}.$$

This complete the proof.

Proof of Theorem 2. By some argument as in proof of (a), we have

$$
\begin{aligned}
&\frac{1}{|Q|}\int_Q|S_\delta^{\vec{b}}(f)(x)-S_\delta(((b_1)_Q-b_1)\cdots((b_m)_Q-b_m)f_2)(x_0)|dx\\
\leq\ &\frac{1}{|Q|}\int_Q I_1(x)dx+\frac{1}{|Q|}\int_Q I_2(x)dx+\frac{1}{|Q|}\int_Q I_3(x)dx+\frac{1}{|Q|}\int_Q I_4(x)dx\\
\leq\ &C||\vec{b}||_{Lip_\beta}(M_{m\beta,1}(S_\delta(f))+M_{\delta+m\beta,r}(f),
\end{aligned}
$$

thus

$$(S_\delta^{\vec{b}}(f))^{\#}\leq C||\vec{b}||_{Lip_\beta}(M_{m\beta,1}(S_\delta(f))+M_{\delta+m\beta,r}(f).$$

By using Lemma 3 and the boundedness of S_δ, we have

$$
\begin{aligned}
||S_\delta^{\vec{b}}(f)||_{L^q}&\leq C||(S_\delta^{\vec{b}}(f))^{\#}||_{L^q}\\
&\leq C||\vec{b}||_{Lip_\beta}(||M_{m\beta,1}(S_\delta(f))||_{L^q}+||M_{\delta+m\beta,r}(f)||_{L^q}\\
&\leq C||f||_{L^p}.
\end{aligned}
$$

This complete the proof.
Proof of Theorem 3. It suffices to show that there exists a constant $C > 0$ such that for every H^p-atom a,

$$||S_\delta^{\vec{b}}(a)||_{L^q} \leq C.$$

Let a be a H^p-atom, that is that a supported on a cube $Q = Q(x_0, r)$, $||a||_{L^\infty} \leq |Q|^{-1/p}$ and $\int_{R^n} a(x)x^\gamma dx = 0$ for $|\gamma| \leq [n(1/p-1)]$.

When $m = 1$ see[10]. Now consider the case $m \geq 2$. Write

$$\begin{aligned} ||S_\delta^{\vec{b}}(a)(x)||_{L^q} &\leq \left(\int_{|x-x_0|\leq 2r} |S_\delta^{\vec{b}}(a)(x)|^q dx\right)^{1/q} + \left(\int_{|x-x_0|>2r} |S_\delta^{\vec{b}}(a)(x)|^q dx\right)^{1/q} \\ &= I + II. \end{aligned}$$

For I, choose $1 < p_1 < m/n\beta$ and q_1 such that $1/q_1 = 1/p_1 - m\beta/n$. By the boundednss of $S_\delta^{\vec{b}}$ from $L^{p_1}(R^n)$ to $L^{q_1}(R^n)$(see **Theorem 1**), we get

$$I \leq C||S_\delta^{\vec{b}}(a)||_{L^{q_1}} r^{n(1/q-1/q_1)} \leq C||a||_{L^{p_1}} r^{n(1/q-1/q_1)} \leq C.$$

For II, let $\tau, \tau' \in N$ such that $\tau + \tau' = m$, and $\tau' \neq 0$. We get

$$\begin{aligned} |F_t^{\vec{b}}(a)(x,y)| &= \left|\int_B \prod_{j=1}^m (b_j(x) - b_j(z))\psi_t(y-z)a(z)dz\right| \\ &\leq \left|(b_1(x) - b_1(x_0)) \cdots (b_m(x) - b_m(x_0)) \int_B (\psi_t(y-z) - \psi_t(y-x_0))a(z)dz\right| \\ &\quad + \sum_{j=1}^m \sum_{\sigma \in C_j^m} \left|(b(x) - b(x_0))_{\sigma^c} \int_B (b(z) - b(x_0))_\sigma \psi_t(y-z)a(z)dz\right| \\ &\leq C||\vec{b}||_{Lip_\beta} |x - x_0|^{m\beta} \cdot \int_B |\psi_t(y-z) - \psi_t(y-x_0)||a(z)|dz \\ &\quad + C||\vec{b}||_{Lip_\beta} \sum_{\tau+\tau'=m} |x - x_0|^{\tau\beta} \int_B |z - x_0|^{\tau'\beta} |\psi_t(y-z)||a(z)|dz \\ &\leq C||\vec{b}||_{Lip_\beta} \frac{|x - x_0|^{m\beta} t}{(t + |y - x_0|)^{n+1+\varepsilon-\delta}} \int_B |x_0 - z|^\varepsilon |a(z)|dz \\ &\quad + C||\vec{b}||_{Lip_\beta} \sum_{\tau+\tau'=m} |x - x_0|^{\tau\beta} \frac{t}{(t + |y - z|)^{n+1-\delta}} \int_B |z - x_0|^{\tau'\beta} |a(z)|dz \\ &\leq C||\vec{b}||_{Lip_\beta} \frac{t}{(t + |y - x_0|)^{n+1+\varepsilon-\delta}} \cdot r^{m\beta+\varepsilon+n(1-1/p)} \\ &\quad + C||\vec{b}||_{Lip_\beta} \frac{t}{(t + |y - x_0|)^{n+1-\delta}} \cdot r^{m\beta+n(1-1/p)}, \end{aligned}$$

thus

$$
\begin{aligned}
|S_\delta^{\vec{b}}(a)(x)| &\leq C||\vec{b}||_{Lip_\beta}\left(\int\int_{\Gamma(x)}\left(\frac{t}{(t+|y-x_0|)^{n+1+\varepsilon-\delta}}\right)^2\frac{dydt}{t^{n+1}}\right)^{1/2}\cdot r^{m\beta+\varepsilon+n(1-1/p)}\\
&\quad +C||\vec{b}||_{Lip_\beta}\left(\int\int_{\Gamma(x)}\left(\frac{t}{(t+|y-x_0|)^{n+1-\delta}}\right)^2\frac{dydt}{t^{n+1}}\right)^{1/2}\cdot r^{m\beta+n(1-1/p)}\\
&\leq C||\vec{b}||_{Lip_\beta}\left(\int\int_{\Gamma(x)}\frac{t^{1-n}2^{2(n+1+\varepsilon-\delta)}}{(2t+2|y-x_0|)^{2(n+1+\varepsilon-\delta)}}dydt\right)^{1/2}\cdot r^{m\beta+\varepsilon+n(1-1/p)}\\
&\quad +C||\vec{b}||_{Lip_\beta}\left(\int\int_{\Gamma(x)}\frac{t^{1-n}2^{2(n+1-\delta)}}{(2t+2|y-x_0|)^{2(n+1-\delta)}}dydt\right)^{1/2}\cdot r^{m\beta+n(1-1/p)},
\end{aligned}
$$

note that $2t+|x_0-y|>2t+|x_0-x|-|x-y|>t+|x_0-x|$ when $|x-y|<t$

$$
\int_0^\infty\frac{tdt}{(t+|x-x_0|)^{2(n+1+\varepsilon-\delta)}}=C|x-x_0|^{-(n+\varepsilon-\delta)};
$$

then we deduce

$$
\begin{aligned}
|S_\delta^{\vec{b}}(a)(x)| &\leq C||\vec{b}||_{Lip_\beta}\left(\int\int_{\Gamma(x)}\frac{t^{1-n}}{(t+|x-x_0|)^{2(n+1+\varepsilon-\delta)}}dydt\right)^{1/2}\cdot r^{m\beta+\varepsilon+n(1-1/p)}\\
&\quad +C||\vec{b}||_{Lip_\beta}\left(\int\int_{\Gamma(x)}\frac{t^{1-n}}{(t+|x-x_0|)^{2(n+1-\delta)}}dydt\right)^{1/2}\cdot\cdot r^{m\beta+\varepsilon+n(1-1/p)}\\
&\leq C||\vec{b}||_{Lip_\beta}\left(\int_0^\infty\frac{tdt}{(t+|x-x_0|)^{2(n+1+\varepsilon-\delta)}}\right)^{1/2}\cdot r^{m\beta+\varepsilon+n(1-1/p)}\\
&\quad +||\vec{b}||_{Lip_\beta}\left(\int_0^\infty\frac{tdt}{(t+|x-x_0|)^{2(n+1-\delta)}}\right)^{1/2}\cdot r^{m\beta+n(1-1/p)}\\
&\leq C||\vec{b}||_{Lip_\beta}|x-x_0|^{-n+\delta}\cdot r^{m\beta+n(1-1/p)},
\end{aligned}
$$

so

$$
\begin{aligned}
II &\leq C||\vec{b}||_{Lip_\beta}\cdot r^{m\beta+n(1-1/p)}\left(\int_{|x-x_0|>2r}|x-x_0|^{(\delta-n)q}dx\right)^{1/q}\\
&\leq C||\vec{b}||_{Lip_\beta}.
\end{aligned}
$$

This complete the proof of Theorem 3.

Proof of Theorem 4. By Lemma 5, let $f\in H\dot{K}_{q_1}^{\alpha,p}(R^n)$ and $f=\sum_{j=-\infty}^{\infty}\lambda_j a_j$, *supp* $a_j \subset B_j = B(0, 2^j)$, a_j be a central $(\alpha, q)-$atom, and $\sum_{j=-\infty}^{\infty} |\lambda_j|^p < \infty$. We have

$$\begin{aligned} ||S_\delta^{\vec{b}}(f)||^p_{\dot{K}^{\alpha,p}_{q_2}} &\le C \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=-\infty}^{k-2} |\lambda_j| ||S_\delta^{\vec{b}}(a_j)\chi_k||_{L^{q_2}} \right)^p \\ &\quad + C \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=k-1}^{\infty} |\lambda_j| ||S_\delta^{\vec{b}}(a_j)\chi_k||_{L^{q_2}} \right)^p \\ &= I + II. \end{aligned}$$

For II, by the boundedness of $S_\delta^{\vec{b}}$ on (L^{q_1}, L^{q_2}), we get

$$\begin{aligned} II &\le C||\vec{b}||^p_{Lip_\beta} \sum_{k=-\infty}^{\infty} 2^{k\alpha p} (\sum_{j=k-1}^{\infty} |\lambda_j| ||a_j||_{L^{q_1}})^p \\ &\le C||\vec{b}||^p_{Lip_\beta} \sum_{k=-\infty}^{\infty} 2^{k\alpha p} (\sum_{j=k-1}^{\infty} |\lambda_j| \cdot 2^{-j\alpha})^p \\ &\le C||\vec{b}||^p_{Lip_\beta} \begin{cases} \sum_{k=-\infty}^{\infty} \sum_{j=k-1}^{\infty} |\lambda_j|^p \cdot 2^{(k-j)\alpha p}, & 0 < p \le 1 \\ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} (\sum_{j=k-1}^{\infty} |\lambda_j|^p \cdot 2^{-j\alpha p/2}) (\sum_{j=k-1}^{\infty} 2^{-j\alpha p'/2})^{p/p'}, & 1 < p < \infty \end{cases} \\ &\le C||\vec{b}||^p_{Lip_\beta} \sum_{j=-\infty}^{\infty} |\lambda_j|^p. \end{aligned}$$

For I, when $m = 1$, similar to Theorem 3, we have

$$\begin{aligned} & |F_t^{b_1}(a_j)(x,y)| \le |(b_1(x) - b_1(0)) \int_{B_j} (\psi_t(y-z) - \psi_t(y)) a_j(z) dz| \\ & + | \int_{B_j} \psi_t (b_1(z) - b_1(0)) a_j(z) dz| \\ \le\ & C||b_1||_{Lip_\beta} \left[\int_{B_j} \frac{|x|^\beta |z|^\varepsilon t}{(t+|y|)^{n+1+\varepsilon-\delta}} \cdot |a_j(z)| dz \right. \\ & \left. + \int_{B_j} \frac{t|z|^\beta}{(t+|y-z|)^{n+1-\delta}} \cdot |a_j(z)| dz \right] \\ \le\ & C||b_1||_{Lip_\beta} \left[\frac{|x|^\beta t}{(t+|y|)^{n+1+\varepsilon-\delta}} \int_{B_j} |z|^\varepsilon |a_j(z)| dz \right. \\ & \left. + \frac{t}{(t+|y|)^{n+1-\delta}} \int_{B_j} |z|^\beta |a_j(z)| dz \right] \\ \le\ & C||b_1||_{Lip_\beta} \left[\frac{|x|^\beta t}{(t+|y|)^{n+1+\varepsilon-\delta}} \cdot 2^{j(\varepsilon + n(1-1/q_1) - \alpha)} + \frac{t}{(t+|y|)^{n+1-\delta}} \cdot 2^{j(\beta + n(1-1/q_1)-\alpha)} \right], \end{aligned}$$

thus

$$\begin{aligned}
S_\delta^{b_1}(a_j)(x) &\leq C||b_1||_{Lip_\beta}\left[\left(\int\int_{\Gamma(x)}\left(\frac{t}{(t+|y|)^{n+1+\varepsilon-\delta}}\right)^2\frac{dydt}{t^{n+1}}\right)^{1/2}\cdot|x|^\beta\cdot 2^{j(\varepsilon+n(1-1/q_1)-\alpha)}\right.\\
&\quad\left.+\left(\int\int_{\Gamma(x)}\left(\frac{t}{(t+|x|)^{n+1-\delta}}\right)^2\frac{dydt}{t^{n+1}}\right)^{1/2}\cdot 2^{j(\beta+n(1-1/q_1)-\alpha)}\right]\\
&\leq C||b_1||_{Lip_\beta}\left[|x|^{-(n+\varepsilon-\delta)}\cdot|x|^\beta\cdot 2^{j(\varepsilon+n(1-1/q_1)-\alpha)}\right.\\
&\quad\left.|x|^{-n+\delta}\cdot 2^{j(\beta+n(1-1/q_1)-\alpha)}\right]\\
&\leq C||b_1||_{Lip_\beta}|x|^{-n+\delta}\cdot 2^{j(\beta+n(1-1/q_1)-\alpha)}.
\end{aligned}$$

From that we have

$$\begin{aligned}
||S_\delta^{b_1}(a_j)\chi_k||_{L^{q_2}} &\leq C||b_1||_{Lip_\beta}\cdot 2^{j(\beta+n(1-1/q_1)-\alpha)}\left(\int_{B_k}|x|^{(\delta-n)q_2}dx\right)^{1/q_2}\\
&\leq C||b_1||_{Lip_\beta}\cdot 2^{j(\beta+n(1-1/q_1)-\alpha)}\cdot 2^{-kn(1-1/q_2)}\\
&\leq C||b_1||_{Lip_\beta}\cdot 2^{[j(\beta+n(1-1/q_1)-\alpha)-k(\beta+n(1-1/q_1))]},
\end{aligned}$$

so

$$\begin{aligned}
I_1 &\leq C||b_1||^p_{Lip_\beta}\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|\cdot 2^{[j(\beta+n(1-1/q_1)-\alpha)-k(\beta+n(1-1/q_1))]}\right)^p\\
&\leq C||b_1||^p_{Lip_\beta}\begin{cases}\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{k-2}|\lambda_j|^p\cdot 2^{(j-k)(\beta+n(1-1/q_1)-\alpha)p}, & 0<p\leq 1\\ \sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|^p\cdot 2^{p[j(\beta+n(1-1/q_1)-\alpha)-k(\beta+n(1-1/q_1))]/2}\right) & \\ \quad\times\left(\sum_{j=-\infty}^{k-2}2^{p'[j(\beta+n(1-1/q_1)-\alpha)-k(\beta+n(1-1q_1))]/2}\right)^{p/p'}, & 1<p<\infty\end{cases}\\
&\leq C||b_1||^p_{Lip_\beta}\begin{cases}\sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=j+2}^{\infty}2^{(j-k)(\beta+n(1-1/q_1)-\alpha)p}, & 0<p\leq 1\\ \sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=j+2}^{\infty}2^{p[(j-k)(\beta+n(1-1/q_1)-\alpha)]/2}, & 1<p<\infty\end{cases}\\
&\leq C||b_1||^p_{Lip_\beta}\sum_{j=-\infty}^{\infty}|\lambda_j|^p.
\end{aligned}$$

Then

$$||S_\delta^{b_1}(f)||_{\dot K_{q_2}^{\alpha,p}}\leq C||b_1||_{Lip_\beta}(\sum_{j=-\infty}^{\infty}|\lambda_j|^p)^{1/p}\leq C||f||_{H\dot K_{q_1}^{\alpha,p}}.$$

When $m\geq 2$, we have

$$|F_t^{\vec{b}}(a_j))(x,y)| \leq |(b_1(x)-b_1(0))\cdots(b_m(x)-b_m(0))\int_{B_j}(\psi_t(y-z)-\psi_t(y))a_j(z)dz|$$

$$
\begin{aligned}
&+\sum_{j=1}^{\infty}\sum_{\sigma\in C_j^m}|(b(x)-b(0))_{\sigma^c}\int_{B_j}(b(z)-b(0))_{\sigma}\psi_t(y-z)a_j(z)dz| \\
\le\ & C||\vec{b}||_{Lip_\beta}|x|^{m\beta}\int_{B_j}|\psi_t(y-z)-\psi_t(y)||a_j(z)|dz \\
&+C||\vec{b}||_{Lip_\beta}\sum_{\tau+\tau'=m}|x|^{\tau\beta}\int_{B_j}|z|^{\tau'\beta}|\psi_t(y-z)||a_j(z)|dz \\
\le\ & C||\vec{b}||_{Lip_\beta}\frac{|x|^{m\beta}t}{(t+|y|)^{n+1+\varepsilon-\delta}}\int_{B_j}|y|^{\varepsilon}|a_j(z)|dz \\
&+C||\vec{b}||_{Lip_\beta}\sum_{\tau+\tau'=m}\frac{|x|^{\tau\beta}t}{(t+|y-z|)^{n+1-\delta}}\int_{B_j}|z|^{\tau'\beta}|a_j(z)|dz \\
\le\ & C||\vec{b}||_{Lip_\beta}\frac{|x|^{m\beta}t}{(t+|y|)^{n+1+\varepsilon-\delta}}\cdot 2^{j(\varepsilon+n(1-1/q_1)-\alpha)} \\
&+C||\vec{b}||_{Lip_\beta}\sum_{\tau+\tau'=m}\frac{|x|^{\tau\beta}t}{(t+|y|)^{n+1-\delta}}\cdot 2^{j(\tau'\beta+n(1-1/q_1)-\alpha)},
\end{aligned}
$$

thus

$$
\begin{aligned}
&S_\delta^{\vec{b}}(a_j)(x)=\left(\int\int_{\Gamma(x)}|F_t^{\vec{b}}(a_j)(x,y)|^2\frac{dydt}{t^{n+1}}\right)^{1/2} \\
\le\ & C||\vec{b}||_{Lip_\beta}|x|^{m\beta}\cdot 2^{j(\varepsilon+n(1-1/q_1)-\alpha)}\cdot\left(\int\int_{\Gamma(x)}\left(\frac{t}{(t+|y|)^{n+1+\varepsilon-\delta}}\right)^2\frac{dydt}{t^{n+1}}\right)^{1/2} \\
&+C||\vec{b}||_{Lip_\beta}\sum_{\tau+\tau'=m}|x|^{\tau\beta}\cdot 2^{j(\tau'\beta+n(1-1/q_1)-\alpha)}\cdot\left(\int\int_{\Gamma(x)}\left(\frac{t}{(t+|y|)^{n+1-\delta}}\right)^2\frac{dydt}{t^{n+1}}\right)^{1/2} \\
\le\ & C||\vec{b}||_{Lip_\beta}|x|^{m\beta}|x|^{-(n+\varepsilon-\delta)}\cdot 2^{j(\varepsilon+n(1-1/q_1)-\alpha)} \\
&+C||\vec{b}||_{Lip_\beta}\sum_{\tau+\tau'=m}|x|^{\tau\beta}|x|^{\delta-n}\cdot 2^{j(\tau'\beta+n(1-1/q_1)-\alpha)} \\
\le\ & C||\vec{b}||_{Lip_\beta}|x|^{\delta-n}\cdot 2^{j(m\beta+n(1-1/q_1)-\alpha)},
\end{aligned}
$$

then

$$
\begin{aligned}
&||S_\delta^{\vec{b}}(a_j)\chi_k||_{L^{q_2}} \\
\le\ & C||\vec{b}||_{Lip_\beta}\cdot 2^{j(m\beta+n(1-1/q_1)-\alpha)}\cdot\left(\int_{B_j}|x|^{(\delta-n)q_2}dx\right)^{1/q_2} \\
\le\ & C||\vec{b}||_{Lip_\beta}\cdot 2^{[j(m\beta+n(1-1/q_1)-\alpha)-k(m\beta+n(1-1/q_1))]},
\end{aligned}
$$

so

$$
\begin{aligned}
I &\leq C||\vec{b}||^p_{Lip_\beta} \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=-\infty}^{k-2} |\lambda_j| \cdot 2^{[j(m\beta+n(1-1/q_1)-\alpha)-k(m\beta+n(1-1/q_1))]} \right)^p \\
&\leq C||\vec{b}||^p_{Lip_\beta} \begin{cases} \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{k-2} |\lambda_j|^p \cdot 2^{(j-k)(m\beta+n(1-1/q_1)-\alpha)p}, & 0 < p \leq 1 \\ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=-\infty}^{k-2} |\lambda_j|^p \cdot 2^{p[j(m\beta+n(1-1/q_1)-\alpha)-k(m\beta+n(1-1/q_1))]/2} \right) & \\ \quad \times \left(\sum_{j=-\infty}^{k-2} 2^{p'[j(m\beta+n(1-1/q_1)-\alpha)-k(m\beta+n(1-1/q_1))]/2} \right)^{p/p'}, & 1 < p < \infty \end{cases} \\
&\leq C||\vec{b}||^p_{Lip_\beta} \sum_{j=-\infty}^{\infty} |\lambda_j|^p.
\end{aligned}
$$

From I and II, we have

$$||S_\delta^{\vec{b}}(f)|| \leq C||\vec{b}||_{Lip_\beta} \left(\sum_{j=-\infty}^{\infty} |\lambda_j|^p \right)^{1/p} \leq C||f||_{H\dot{K}_{q_1}^{\alpha,p}}.$$

This completes the proof of Theorem 4.

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