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SUBORDINATION RESULTS DEFINED BY A NEW DIFFERENTIAL OPERATOR

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Abstract. In this article, we study the differential subordination for certain subclass of functions defined by a new differential operator.

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1. Introduction and Definitions

Let U denote the class of analytic functions in the unit disk $U = \{z : |z| < 1\}$ and $U^* = U - \{0\}$. We can let

$$A(n) = \{f \in H(U), f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + ..., z \in U\}$$

with $A(1) = A$. Let ℓ_n denote the class of functions in U^* of the form

$$f(z) = z + \Sigma_{k=n+1}^{\infty} a_k z^k, n \in N = \{1, 2, ...\}.$$

Let f, g be analytic functions in U. We say that f is subordinate to g, if there exists a Schwarz function $w(z)$, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$, such that $f(z) = g(w(z))$, $(z \in U)$, and symbolically written as the following: $f \prec g(z \in U)$ *or* $f(z) \prec g(z)(z \in U)$. It is known that $f(z) \prec g(z)$ $(z \in U) \Rightarrow f(0) = g(0)$ and $f(U) \subset g(U)$. Further, if the function g is univalent in U, then we have the following equivalent

$$f(z) \prec g(z) \quad (z \in U) \quad \Leftrightarrow f(0) = g(0) \quad and \quad f(U) \prec g(z).$$

A function$f \in H(U)$ is said to be convex if it is univalent and $f(U)$ is a convex domain. It is well known that the function f is convex if and only if $f'(0) \neq 0$ and

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0(z \in U).$$

We denote all this class of functions by K.

Definition 1. *Let the function f be in the class A_n. For $m,\alpha \in N_0 = N \cup \{0\}$, $\lambda_2 \geq \lambda_1 \geq 0$, we define the following differential operator*

$$D^{m,\alpha}_{\lambda_1,\lambda_2} f(z) = z + \sum_{k=n+1}^{\infty} \left[\frac{1 + (\lambda_1 + \lambda_2)(k-1)}{1 + \lambda_2(k-1)} \right]^m C(\alpha, k) a_k z^k. \tag{1}$$

Proposition 1. *For $m,\alpha \in N_0$, $\lambda_2 \geq \lambda_1 \geq 0$*

$$(1 + \lambda_2(k-1)) D^{m+1}(\lambda_1,\lambda_2,\alpha) f(z)$$
$$= (1 + \lambda_2(k-1) - \lambda_1) D^m(\lambda_1,\lambda_2,\alpha) f(z) + \lambda_1 z (D^m(\lambda_1,\lambda_2,\alpha) f(z))' \tag{2}$$

and

$$D^{m_1}(\lambda_1,\lambda_2,\alpha)(D^{m_2}(\lambda_1,\lambda_2,\alpha)) f(z) = D^{m_1+m_2}(\lambda_1,\lambda_2,\alpha)$$
$$= D^{m_2}(\lambda_1,\lambda_2,\alpha)(D^{m_1}(\lambda_1,\lambda_2,\alpha) f(z), \text{for all integers}\, m_1,\ m_2. \tag{3}$$

Special cases of this operator includes the Ruscheweyh derivative operator in the case $D^0(\lambda_1,\lambda_2,\alpha) \equiv R^n$ [4], the Salagean derivative operator in the case $D^m(1,0,0) \equiv D^m \equiv S^n$ [5], the generalized Salagean derivative operator introduced by Al-Oboudi in the case $D^m(\lambda_1, 0, 0) \equiv D^m_{\lambda_1}$ [1], the generalized Ruscheweyh derivative operator,in the case $D^1(\lambda_1, 0, \alpha) \equiv D^{\lambda_1}_{\alpha}$ [2];the generalized Al-Shaqsi and Darus derivative operator in the case $D^m(\lambda_1, 0, \alpha) \equiv D^{m,\lambda_1}_{\alpha}$ [6].

Also if $f \in A(n)$, then we can write

$$D^m(\lambda_1,\lambda_2,\alpha) f(z) = (f * \wp^{m,\lambda_1}_{\lambda_2,\alpha})(z),$$

$$\wp^{m,\lambda_1}_{\lambda_2,\alpha}(z) = z + \sum_{k=n+1}^{\infty} \left[\frac{1 + (\lambda_1 + \lambda_2)(k-1)}{1 + \lambda_2(k-1)} \right]^m C(\alpha, k) z^k \tag{4}$$

To prove our main results, we shall need the following lemmas.

Lemma 1.[3] *Let the function $h(z)$ be analytic and convex (univalent) in U whit $h(0) = 1$. Assume also the function $\wp(z)$ given by*

$$\wp(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \ldots\ldots \tag{5}$$

be analytic in U. If $\wp(z) + \frac{z\wp'(z)}{\delta} < h(z)$ $\{Re(\delta) \geq 0; \delta \neq 0, z \in U\}$ then

$$\wp(z) < \psi(z) = \frac{\delta}{n} z^{-(\frac{\delta}{n})} \int_0^z t^{(\frac{\delta}{n})-1} h(t) dt < h(z) \quad (z \in U) \tag{6}$$

and ψ is the best dominant.

Lemma 2.[3] *Let $f \in A$,$\delta > 1$ and F is given by*

$$F(z) = \frac{1+\delta}{\delta z^{\frac{1}{\delta}}} \int_0^z f(t) t^{\frac{1}{\delta}-1} dt.$$

If

$$Re\{1 + \frac{zf''(z)}{f'(z)}\} > -\frac{1}{2} \quad (z \in U).$$

Then F is convex.

2. Main result

Now we suppose throughout this paper that $m \in N_0$, $p, n \in N$, $\lambda_2 \geq \lambda_1 > 0$.

Theorem 1. *Let $h \in H(U)$, with $h(0) = 1$ which verifies the inequality:*

$$Re\left[1 + \frac{zh''(z)}{h'(z)}\right] > -\frac{\frac{(\lambda_2(k-1)+1)}{\lambda_1}}{2(p+k)} \quad (z \in U) \tag{7}$$

If $f \in \ell_n$ and verifies the differential subordination

$$[D^{m+1}(\lambda_1,\lambda_2,\alpha) f(z)]' < h(z) \quad (z \in U) \tag{8}$$

then

$$[D^{m}(\lambda_1,\lambda_2,\alpha) f(z)]' < g(z) \quad (z \in U)]$$

where

$$g(z) = \frac{(\frac{\lambda_2(k-1)+1}{\lambda_1})}{(p+k) z^{\frac{(\frac{\lambda_2(k-1)+1}{\lambda_1})}{p+k}}} \int_0^z h(t) t^{\frac{(\frac{\lambda_2(k-1)+1}{\lambda_1})}{p+k}-1} dt \tag{9}$$

the function g is convex and is the best $(1, p+k)$ dominant.

Proof: From the identity(2) we have

$$D^{m+1}(\lambda_1,\lambda_2,\alpha) f(z) = \left(1 - \frac{\lambda_1}{\lambda_2(k-1)+1}\right) \quad D^{m}(\lambda_1,\lambda_2,\alpha) f(z)$$

$$+ \left(\frac{\lambda_1}{\lambda_2(k-1)+1}\right) z(D^{m}(\lambda_1,\lambda_2,\alpha) f(z))' \tag{10}$$

differentiating (10) with respect to z,we obtain

$$D^{m+1}(\lambda_1,\lambda_2,\alpha) f(z) = \frac{\lambda_1}{\lambda_2(k-1)+1}$$

$$\left[z(D^m,(\lambda_1,\lambda_2,\alpha)f(z))'' + \frac{\lambda_2(k-1)+1}{\lambda_1}(D^m(\lambda_1,\lambda_2,\alpha)f(z))'\right](\lambda_1, z \in U) \quad (11)$$

If we let

$$q(z) = [(D^m(\lambda_1,\lambda_2,\alpha)f(z)]' \quad (z \in U) \quad (12)$$

then (11) becomes

$$[D^{m+1}(\lambda_1,\lambda_2,\alpha)f(z)]' = q(z) + \left(\frac{\lambda_1}{\lambda_2(k-1)+1}\right) zq'(z) \prec (z) \quad (13)$$

Using (13), subordination (8) is equivalent to

$$q(z) + \frac{\lambda_1}{\lambda_2(k-1)+1} \quad zq'(z) \prec h(z) \quad (z \in U), \quad (14)$$

where

$$q(z) = 1 + c_{p+k+1}z^{p+k} + \dots.$$

By using Lemma 1 for $\delta = \frac{\lambda_2(k-1)+1}{\lambda_1}$, $n = p + k$, we have

$$g(z) = \frac{(\frac{\lambda_2(k-1)+1}{\lambda_1})}{(p+k)z^{\frac{(\frac{\lambda_2(k-1)+1}{\lambda_1})}{p+k}}} \int_0^z h(t)t^{\frac{(\frac{\lambda_2(k-1)+1}{\lambda_1})}{p+k}-1}dt$$

is the best dominant.

By applying Lemma 2 for the function given by (9) and function h with the property in (7) for $\delta = \frac{\lambda_2(k-1)+1}{\lambda_1}$, we observed that the function g is convex.

As a consequence of Theorem 1, we have the following corollary. Put $p = \lambda_1 = 1$ and $m = k = \lambda_2 = \alpha = 0$ in Theorem 1 we have

Corollary 1. *Let $h \in H(U)$, with $h(0) = 1$ which satisfies the in equality*

$$Re\left\{1 + \frac{zh''(z)}{h'(z)}\right\} > -\frac{1}{2} \quad (z \in U).$$

If $f \in \ell_n$ and satisfies the the differential subordination: $zf''(z)+f'(z) < h(z) \quad (z \in U)$, then $f'(z) < g(z) \quad (z \in U)$, where $g(z) = \frac{1}{z}\int_0^z h(t)dt \quad (z \in U)$. The function g is convex and is the best dominant.

Theorem 2. *Let $h \in H(U)$ with $h(0) = 1$ which satisfy the inequality.*

$$Re\left\{1 + \frac{zh''(z)}{h'(z)}\right\} > -\frac{1}{2(p+k)}. \quad (15)$$

If $f \in \ell_n$ and satisfy the differential subordination

$$[D^m(\lambda_1, \lambda_2, \delta)f(z)]' \prec h(z) \quad (z \in U). \tag{16}$$

Then

$$\frac{D^m(\lambda_1, \lambda_2, \delta)f(z)}{z} \prec g(z), \quad (z \in U),$$

where

$$g(z) = \frac{1}{(p+k)z^{\left(\frac{1}{p+k}\right)}} \int_0^z h(t)t^{\left(\frac{1}{p+k}\right)-1}dt \quad (z \in U) \tag{17}$$

Proof: We let

$$q(z) = \frac{D^m(\lambda_1, \lambda_2, \delta)f(z)}{z} \quad (z \in U) \tag{18}$$

and obtain

$$(zq'(z)) + q(z) = (D^m(\lambda_1, \lambda_2, \delta)f(z))'.$$

Then (15) gives

$$q(z) + zq'(z) \prec h(z)$$

where

$$q(z) = 1 + q_{p+k+1}z^{p+k} + ... (z \in U).$$

By using Lemma 1 for $\delta = 1, n = p + k$, we have

$$q(z) \prec g(z) \prec h(z)$$

where

$$\frac{1}{(p+k)z^{\left(\frac{1}{p+k}\right)}} \int_0^z h(t)t^{\left(\frac{1}{p+k}\right)-1}dt \quad (z \in U).$$

This is the best dominant.

By applying Lemma 2 for function g given by (17) and for function h with the property in (15) for $n = p + k$, we observed that the function g is convex.

As a consequence of Theorem 2 and by choosing $p = \lambda_1 = 1$ and $m = k = \lambda_2 = \alpha = 0$, we have the following interesting corollary.

Corollary 2. *Let $h \in H(U)$, with $h(0) = 1$ satisfying the differential subordination*

$$Re\left\{1 + \frac{zh''(z)}{h'(z)}\right\} > -\frac{1}{2} \quad (z \in U).$$

If $f \in \ell_n$ and satisfies the differential subordination $f'(z) \prec h(z) \quad (z \in U)$, then $\frac{f(z)}{z} \prec g(z)$ where $g(z) = \frac{1}{z}\int_0^z h(t)dt$, $(z \in U)$.

Various other studies related to differential operators for different type of classes can also be found in the following articles (see for examples [7]-[9]).

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