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# A METHOD FOR SOLVING FULLY FUZZY QUADRATIC PROGRAMMING PROBLEMS

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Abstract. In this paper we focus on a kind of quadratic programming with fuzzy numbers and variables. First by using a fuzzy ranking and arithmetic operations, we transform these problems to crisp model with non-linear objective and linear constraints, then by solving this problem we obtain a fuzzy optimal solution. Finally, we give an illustrative example.

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## 1. Introduction

In fuzzy decision making problems, the concept of maximizing was proposed by Bellman and Zadeh [3]. This concept was adopted to problems of mathematical programming by Tanaka et al.[10]. Quadratic programming is a mathematical programming designed to optimize the usage of limited resources. It has led to a number of interesting applications and the development of numerous useful results [1, 2, 8, 9]. Quadratic programming is one of the most important optimization techniques in operations research. In real-world applications, quadratic programming models usually are formulated to find some future course of action. The parameter values used would be based on a prediction of future conditions which inevitably involves some degree of uncertainty. If some parameters are imprecise or uncertain, then some crisp values are usually assigned to those uncertain parameters to make the conventional quadratic program workable. The research on fuzzy mathematical programming has been an active area since Bellman and Zadeh proposed the definition of fuzzy decision [3, 4, 6, 13]. As the definition of Bellman and Zadeh, fuzzy decision may be described as the best balance degree of fuzzy objective and resource constraints. Based on Zadeh's extension principle [12, 14], the fuzzy quadratic programming problem is transformed into a pair of two-level mathematical programs to calculate the upper and lower bounds of the objective value at possibility level $\alpha$. The membership function of the fuzzy objective value is derived numerically by enumerating different values of $\alpha$.

In this paper, we consider fuzzy quadratic programming problems in which all the parameters as well as the variables are fuzzy numbers and is known as Fully Fuzzy Quadratic Programming (FFQP) problems. We invoke the Ranking Function to define of a quadratic programming problem with uncertainties in the coefficients, the variables and fuzzy order relation of the set of inequalities constraints and the objective function. Under these settings, we propose a computable method to determine the fuzzy optimal solution.

The paper is organized as follows: In section 2, some preliminary summaries on the fuzzy numbers, the ranking functions and arithmetic operations is introduced. In section 3, we introduced fully fuzzy quadratic programming problem with the fuzzy order relation in the inequalities constraints and the objective function. In section 4, we transformed the FFQP problem to a crisp model with non-linear gaol and linear constraints. Finally in section 5 a numerical example is proposed for illustrate of results.

## 2. Preliminaries

In this section, we review some necessary notations and definitions of the fuzzy set theory in which will be used in this paper

### 2.1. Fuzzy numbers

As usual, the fuzzy set $\mu_{\tilde{a}} : R \longrightarrow [0,1]$ is a fuzzy number if

1. $\mu_{\tilde{a}}$ is upper semi-continuous,
2. $Supp(\tilde{a}) = \{x \in R : \ \mu_{\tilde{a}}(x) > 0\}$ is a bounded set on $R$.

The set of all these fuzzy numbers is denoted by $F(R)$ and the function $\mu_{\tilde{a}}(x)$ is called the membership function.

**Definition 1.** *A fuzzy number $\tilde{a}$ is called a convex fuzzy set if* $\mu_{\tilde{a}}(\lambda x + (1-\lambda)y) \geq \min\{\mu_{\tilde{a}}(x), \mu_{\tilde{a}}(y)\}, \ \forall x, y \in R$ and $\lambda \in [0,1]$.

**Definition 2.** *A fuzzy number $\tilde{a}$ is said to be a triangular fuzzy number if its membership function is given by*

$$\mu_{\tilde{a}}(x) = \begin{cases} \frac{x-a}{b-a}, & a \leq x \leq b; \\ \frac{x-c}{b-c}, & b \leq x \leq c; \\ 0, & \text{otherwise}, \end{cases}$$

*which it is parameterized by $\tilde{a} = (a, b, c)$ where $a, c$ are called the lower and upper limits of support of $\tilde{a}$ and $b$ is called the pick value of $\tilde{a}$. Triangular fuzzy numbers are very often used in different applications, such as fuzzy controls, managerial decision* *making, business and finance.*

**Definition 3.** *A triangular fuzzy number* $\tilde{a} = (a, b, c)$ *is called non-negative if* $a \geq 0$.

**Definition 4.**[5] *Two triangular fuzzy numbers* $\tilde{a}_1 = (a_1, b_1, c_1)$ *and* $\tilde{a}_2 = (a_2, b_2, c_2)$ *is called equal if and only if* $a_1 = a_2, b_1 = b_2$ *and* $c_1 = c_2$.

### 2.2. Ranking function

One of the ways for solving mathematical programming problems in a fuzzy environment is to compare fuzzy numbers. The comparison between fuzzy numbers is done by using a ranking function that attends some conditions described in [11]. An appropriate approach for ordering the elements of $F(R)$ is to define a ranking function $\mathcal{R} : F(R) \to R$, which maps each fuzzy number into the real line, where a natural order exists. Let $\tilde{a} = (a, b, c)$ be a triangular fuzzy number, then a special form of the ranking function was first proposed by Liou et al. [7]:

$$\mathcal{R}(\tilde{a}) = \frac{a + 2b + c}{4}. \tag{1}$$

Some order on $F(R)$ is defined as follows: $\tilde{a} \leq^f \tilde{b}$ if and only if $\mathcal{R}(\tilde{a}) \leq \mathcal{R}(\tilde{b})$, where $\tilde{a}$ and $\tilde{b}$ belong to $F(R)$, $\mathcal{R}$ is a ranking function, and the symbol " $\leq^f$ " represents the fuzzy order relation.

### 2.3. Arithmetic operations

In this part, we introduce some arithmetic operations between two triangular fuzzy numbers [5]. Let $\tilde{a}_1 = (a_1, b_1, c_1)$ and $\tilde{a}_2 = (a_2, b_2, c_2)$ be two triangular fuzzy numbers then

1. $\tilde{a}_1 \oplus \tilde{a}_2 = (a_1, b_1, c_1) \oplus (a_2, b_2, c_2) = (a_1 + a_2, b_1 + b_2, c_1 + c_2)$,
2. $-\tilde{a}_1 = (-c_1, -b_1, -a_1)$,
3. Let $\tilde{a} = (a, b, c)$ be any triangular fuzzy number and $\tilde{x} = (x, y, z)$ be a non-negative triangular fuzzy number then

$$\tilde{a} \otimes \tilde{x} = \begin{cases} (ax, by, cz), & a \geq 0; \\ (az, by, cz), & a < 0, c \geq 0; \\ (az, by, cx), & c < 0. \end{cases} \tag{2}$$

## 3. Fully fuzzy quadratic programming problem

Quadratic programming is one of the most important optimization techniques in operations research. In real-world applications, quadratic programming models usually are formulated to find some future course of action. In the real life problems there may exists uncertainly about the parameters. In such a situation the parameters of quadratic programming problems may be represented as fuzzy numbers. We consider a quadratic programming problem with fuzzy numbers and fuzzy variables as follows:

$$\begin{aligned} \min \ & \sum_{j=1}^{n} \tilde{c}_j \otimes \tilde{x}_j \oplus \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} \tilde{x}_k \otimes \tilde{q}_{kj} \otimes \tilde{x}_j \\ s.t. \ & \sum_{j=1}^{n} \tilde{a}_{ij} \otimes \tilde{x}_j \leq^f \tilde{b}_i, \quad i = 1, 2, \ldots, m_1 \\ & \sum_{j=1}^{n} \tilde{a}_{ij} \otimes \tilde{x}_j = \tilde{b}_i, \quad i = m_1 + 1, \ldots, m \\ & \tilde{x}_j, j = 1, 2, \ldots, n, \ \text{are non} - \text{negative triangular fuzzy numbers} \end{aligned} \tag{3}$$

where $\tilde{a}_{ij}, \tilde{b}_i, \tilde{c}_j, \tilde{x}_j \in F(R)$.

**Definition 5.** *A non-negative triangular fuzzy vector $\tilde{x}$ is said to be a fuzzy feasible solution for (3) if $\tilde{x}$ satisfies the constraints*

$$\sum_{j=1}^{n} \tilde{a}_{ij} \otimes \tilde{x}_j \leq^f \tilde{b}_i, \quad i = 1, 2, \ldots, m_1$$
$$\sum_{j=1}^{n} \tilde{a}_{ij} \otimes \tilde{x}_j = \tilde{b}_i, \quad i = m_1 + 1, \ldots, m$$

**Definition 6.** *A fuzzy feasible solution $\tilde{x}^*$ is called a fuzzy optimal solution for (3) if for all fuzzy feasible solutions $\tilde{x}$, we have*

$$\sum_{j=1}^{n} \tilde{c}_j \otimes \tilde{x}_j^* \oplus \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} \tilde{x}_k^* \otimes \tilde{q}_{kj} \otimes \tilde{x}_j^* \leq^f \sum_{j=1}^{n} \tilde{c}_j \otimes \tilde{x}_j \oplus \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} \tilde{x}_k \otimes \tilde{q}_{kj} \otimes \tilde{x}_j.$$

## 4. The solution procedure

In this section, we proposed a method to find of the optimal solution of the problem (3). For this work, we perform the following steps:

**Step 1.** Let the parameters $\tilde{a}_{ij}, \tilde{c}_j, \tilde{b}_i, \tilde{q}_{kj}$ and $\tilde{x}_j$ are represented by triangular fuzzy numbers $(a_{ij}, \alpha_{ij}, \beta_{ij}), (c_j, \alpha_j, \beta_j), (b_i, g_i, h_i), (q_{kj}, p_{kj}, r_{kj})$ and $(x_j, y_j, z_j)$ respectively, in this way the problem (3) may be written as follows:

$$
\begin{aligned}
\min \sum_{j=1}^{n} (c_j, \alpha_j, \beta_j) \otimes (x_j, y_j, z_j) & \\
& \oplus \tfrac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} (x_k, y_k, z_k) \otimes (q_{kj}, p_{kj}, r_{kj}) \otimes (x_j, y_j, z_j) \\
s.t. \ \sum_{j=1}^{n} (a_{ij}, \alpha_{ij}, \beta_{ij}) \otimes (x_j, y_j, z_j) \leq^{f} (b_i, g_i, h_i), & \quad i = 1, 2, \ldots, m_1, \\
\sum_{j=1}^{n} (a_{ij}, \alpha_{ij}, \beta_{ij}) \otimes (x_j, y_j, z_j) = (b_i, g_i, h_i), & \quad i = m_1 + 1, \ldots, m, \\
(x_j, y_j, z_j), j = 1, 2, \ldots, n, \ \text{are non} - \text{negative triangular fuzzy numbers.} &
\end{aligned}
$$

**Step 2.** By using the relation (2), assume that

$$(a_{ij}, \alpha_{ij}, \beta_{ij}) \otimes (x_j, y_j, z_j) = (t_{ij}, u_{ij}, v_{ij}), (c_j, \alpha_j, \beta_j) \otimes (x_j, y_j, z_j) = (t_j, u_j, v_j),$$

and $(x_k, y_k, z_k) \otimes (q_{kj}, p_{kj}, r_{kj}) \otimes (x_j, y_j, z_j) = (t'_{kj}, u'_{kj}, v'_{kj})$, where $t' = f(x, y, z), u' = g(x, y, z)$ and $v' = h(x, y, z)$. In this case, the problem of obtained in Step 1, may be rewritten as:

$$
\begin{aligned}
& \min \sum_{j=1}^{n} (t_j, u_j, v_j) \oplus \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} (t'_{kj}, u'_{kj}, v'_{kj}) \\
& s.t. \ \sum_{j=1}^{n} (t_{ij}, u_{ij}, v_{ij}) \leq^{f} (b_i, g_i, h_i), \quad i = 1, 2, \ldots, m_1, \\
& \sum_{j=1}^{n} (t_{ij}, u_{ij}, v_{ij}) = (b_i, g_i, h_i), \quad i = m_1 + 1, \ldots, m, , \\
& y_j - x_j \geq 0, z_j - y_j \geq 0, j = 1, 2, \ldots, n.
\end{aligned}
$$

**Step 3.** Using the definition 4 and the order relation, defined in subsection 2.2, the problem of obtained in Step 2 may be converted into the following nonlinear programming problem

$$
\begin{aligned}
\min \sum_{j=1}^{n} \frac{t_j + 2u_j + v_j}{4} + \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{t'_{kj} + 2u'_{kj} + v'_{kj}}{4} \\
s.t. \ \sum_{j=1}^{n} \frac{t_{ij} + 2u_{ij} + v_{ij}}{4} \leq \frac{b_i + g_i + h_i}{4}, \quad i = 1, 2, \ldots, m_1, \\
\sum_{j=1}^{n} t_{ij} = b_i, i = m_1 + 1, \ldots, m, \\
\sum_{j=1}^{n} u_{ij} = g_i, i = m_1 + 1, \ldots, m, \\
\sum_{j=1}^{n} v_{ij} = h_i, i = m_1 + 1, \ldots, m, \\
y_j - x_j \geq 0, z_j - y_j \geq 0, j = 1, 2, \ldots, n.
\end{aligned}
$$

**Step 4.** Find the optimal solution $x_j, y_j$ and $z_j$ by solving the nonlinear programming problem of obtained in Step 3.

**Step 5.** Find the fuzzy optimal solution by putting the values of $x_j, y_j$ and $z_j$ in $\tilde{x}_j = (x_j, y_j, z_j)$.

## 5. Numerical example

In this section, examples are presented to illustrate the proposed method.

**Example 1.** *Consider the fully fuzzy quadratic programming problem as follows*

$$
\begin{aligned}
\min \ & (-3, -2, -1) \otimes \tilde{x}_1 \oplus (-9, -6, -3) \otimes \tilde{x}_2 \\
& \oplus \tfrac{1}{2}(\tilde{x}_1, \tilde{x}_2) \otimes \begin{pmatrix} (0, 1, 2) & (-4, -2, 4) \\ (-4, -2, 4) & (-4, 3, 6) \end{pmatrix} \otimes \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} \\
s.t. \ & (0, 1, 2) \otimes \tilde{x}_1 \oplus (-1, 1, 3) \otimes \tilde{x}_2 \leq^f (-2, 3, 4), \\
& (-4, -2, 4) \otimes \tilde{x}_1 \oplus (1, 2, 3) \otimes \tilde{x}_2 \leq^f (-2, 3, 4), \\
& (0, 2, 4) \otimes \tilde{x}_1 \oplus (1, 1, 1) \otimes \tilde{x}_2 \leq^f (1, 3, 5), \\
& \tilde{x}_1, \tilde{x}_2 \in F(R) \ \text{ and are non}-\text{negative.}
\end{aligned}
$$

*We want to solve the problem by using the proposed method. Let* $\tilde{x}_1 = (x_1, y_1, z_1)$ *and* $\tilde{x}_2 = (x_2, y_2, z_2)$*, then*

**Step 1.** *The problem may be written as*

$$\min\ (-3,-2,-1)\otimes(x_1,y_1,z_1)\oplus(-9,-6,-3)\otimes(x_2,y_2,z_2)$$
$$\oplus\ \tfrac{1}{2}((x_1,y_1,z_1),(x_2,y_2,z_2))\otimes\begin{pmatrix}(0,1,2) & (-4,-2,4)\\ (-4,-2,4) & (-4,3,6)\end{pmatrix}\otimes\begin{pmatrix}(x_1,y_1,z_1)\\ (x_2,y_2,z_2)\end{pmatrix}$$
$$s.t.\quad (0,1,2)\otimes(x_1,y_1,z_1)\oplus(-1,1,3)\otimes(x_2,y_2,z_2)\leq^f(-2,3,4),$$
$$(-4,-2,4)\otimes(x_1,y_1,z_1)\oplus(1,2,3)\otimes(x_2,y_2,z_2)\leq^f(-2,3,4),$$
$$(0,2,4)\otimes(x_1,y_1,z_1)\oplus(1,1,1)\otimes(x_2,y_2,z_2)\leq^f(1,3,5),$$
$$(x_1,y_1,z_1),(x_2,y_2,z_2)\in F(R)$$

*and are non-negative.*

**Step 2.** *Using the arithmetic operations, the above problem reduces to the following problem*

$$\min\ (-3z_1-9z_2,-2y_1-6y_2,-x_1-3x_2)$$
$$+\tfrac{1}{2}\Big(-4z_2x_1-4z_1x_2-4z_2x_2,\ y_1^2-4y_1y_2+3y_2^2,\ 2z_1^2+8z_1z_2+6z_2^2\Big)$$
$$s.t.\ (-z_2,y_1+y_2,2z_1+3z_2)\leq^f(-2,3,4),$$
$$(-4z_1+x_2,-2y_1+2y_2,4z_1+3z_2)\leq^f(-2,3,4),$$
$$(x_2,2y_1+y_2,4z_1+z_2)\leq^f(1,3,5),$$
$$y_1-x_1\geq 0,z_1-y_1\geq 0,y_2-x_2\geq 0,z_2-y_2\geq 0.$$

**Step 3.** *Using Step 3 of the proposed algorithm the above problem may be rewritten as follows*

$$\min\ \tfrac{-3z_1-9z_2-4y_1-12y_2-x_1-3x_2}{4}+$$
$$\tfrac{1}{8}\Big[-4z_2x_1-4z_1x_2-4z_2x_2+2y_1^2-8y_1y_2+6y_2^2+2z_1^2+8z_1z_2+6z_2^2\Big]$$
$$s.t.\ 2y_1+2y_2+2z_1+2z_2\leq 8,$$
$$x_2-4y_1+4y_2+3z_2\leq 8,x_2+4y_1+2y_2+4z_1+z_2\leq 12,y_1-x_1\geq 0,$$
$$z_1-y_1\geq 0,y_2-x_2\geq 0,z_2-y_2\geq 0.$$

**Step 4.** *Using the classical produces of non-linear programming is optimal solution*

$$x_1=0.6462,\qquad x_2=1.2308,$$
$$y_1=0.6462,\qquad y_2=1.2308,$$
$$z_1=0.6462,\qquad z_2=1.4769.$$

**Step 5.** *Fuzzy optimal solution is*

$$\tilde{x}_1 = (0.6462, 0.6462, 0.6462), \qquad \tilde{x}_2 = (1.2308, 1.2308, 1.4769).$$

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