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COEFFICIENT ESTIMATES FOR STARLIKE FUNCTIONS OF ORDER β

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Abstract. In this paper, we consider the subclass of starlike functions of order β denoted by $SL^*(\beta)$ and determine the coefficient estimates for this subclass. In addition, the Fekete -Szegö functional $|a_3 - \mu a_2^2|$for this class is obtained when μ is real.

1. Introduction

Let H denote the class of functions of the form

$$f(z) = z + \Sigma_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$ on the complex plane $\mathbb{C}$. Robertson introduced in [6] the class $S^*(\beta)$ of starlike functions of order $\beta \leq 1$, which is defined by $S^*(\beta) = \left\{ f \in A : Re[\frac{zf'(z)}{f(z)}] > \beta \quad ,(z \in \mathbb{U}) \right\}$. If $(0 \leq \beta < 1)$, then a function in either of this set is univalent, if $\beta < 0$ it may fail to be univalent. If f and g are analytic functions in $\mathbb{U}$. Then the function f is said to be subordinate to g, and can be written as $f \prec g$ and $f(z) \prec g(z)$ $(z \in \mathbb{U})$ if and only if there exists the Schwarz function w, analytic in $\mathbb{U}$ such that $w(0) = 0$, $|w(z)| < 1$ for $|z| < 1$ and $f(z) = g(w(z))$. Furthermore, if g is univalent in $\mathbb{U}$ we have the following equivalence $f \prec g \Leftrightarrow f(0) = g(0)$ and $f(u) \subseteq g(u)$. The class $SS^*(\beta)$ of strongly starlike functions of order β
$SS^*(\beta) = \{f \in A : |Arg\frac{zf'(z)}{f(z)}| < \frac{\beta\pi}{2}\}, 0 < \beta \leq 1$, which was introduced in [4]. Moreover, $K - ST \subset SL^*$, for $K = 2 + \sqrt{2}$, where $K - ST$ is the class of k-starlike functions introduced in [5], such that $K - ST := \{f \in A : Re[\frac{zf'(z)}{f(z)}] > K|\frac{zf'(z)}{f(z)} - 1|\}, K \geq 0$. Let us consider $Q(f, z) = \frac{zf'(z)}{f(z)}$. In this way many interesting classes of analytic functions have been defined (see for instance [1]). In this paper we consider the class $SL^*(\beta)$ such that $SL^*(\beta) = \{f \in A : |Q^2(f, z) - (1-\beta)| < 1-\beta\}$. It is easy to see that $f \in SL^*(\beta)$ if and only if $\frac{zf'(z)}{f(z)} \prec q_0(z) = \sqrt{(1-\beta)(1+z)}, q_0(0) = 1-\beta$.

Theorem 1 *[1] The function f belongs to the class $SL^*(\beta)$ if and only if there exists an analytic function $q \in H, q(0) = 0, q(z) \prec q_0(z) = \sqrt{(1-\beta)(1+z)}$,$q_0(0) = 1-\beta$ such that $f(z) = z \exp \int_0^z \frac{q(t)-1}{t}\, dt$.*

In [1], the authors set $q_1 = \frac{3+2z}{3+z}$, $q_2 = \frac{5+3z}{5+z}$, $q_3 = \frac{8+4z}{8+z}$, $q_4 = \frac{9+5z}{9+z}$ and since $q_i \prec q_0$ for $i = 1,2,3,4$, then by (1), the functions

$$f_1(z) = z + \frac{z^2}{3}, \quad f_2(z) = z(1+\frac{z}{5})^2, \quad f_3(z) = z(1+\frac{z}{8})^3, \quad f_4(z) = z(1+\frac{z}{9})^4$$

are in $SL^*(\beta)$.

2. Main results

Theorem 2 *If the function $f(z) = z + a_2z^2 + \cdots$ belongs to the class $SL^*(\beta)$,then $\sum_{k=2}^{\infty}(k^2 - 2(1-\beta))|a_k|^2 \leq 1-\beta$.*

Proof. If $f \in SL^*(\beta)$, then $Q(f,z) < q_0(z) = \sqrt{(1-\beta)(1+z)}$. Let $Q(f,z) = \sqrt{(1-\beta)(1+w(z))}$, where $Q(f,z) = \frac{zf'(z)}{f(z)}$ and w satisfies $w(0) = 0, |w(z)| < 1$ for $|z| < 1$, then $1-\beta)f^2(z) = z^2f'^2(z) - (1-\beta)f^2(z)w(z)$. And using this, we can obtain

$$\begin{aligned} 2\pi(1-\beta)\Sigma_{k=1}^{\infty}|a_k|^2r^{2k} &= (1-\beta)\int_0^{2\pi}|f(re^{i\theta})|^2d\theta \\ &\geq (1-\beta)\int_0^{2\pi}|(f^2(re^{i\theta})w(re^{i\theta})|d\theta \\ &= \int_0^{2\pi}|re^{i\theta}f'(re^{i\theta})|^2 - (1-\beta)|f^2(re^{i\theta})| \\ &= (2\pi\Sigma_{k=1}^{\infty}k^2|a_k|^2r^{2k}) + (1-\beta)(2\pi\Sigma_{k=1}^{\infty}|a_k|^2r^{2k}). \end{aligned}$$

For $0 < r < 1$. The extreme in this sequence of inequality gives $\Sigma_{k=1}^{\infty}(k^2 - (1-\beta)|a_k|^2r^{2k} - (1-\beta)\Sigma_{k=1}^{\infty}|a_k|^2r^{2k} \leq 0$. Eventually, if we let $r \to 1^-$, then $\Sigma_{k=1}^{\infty}(k^2 - 2(1-\beta)|a_k|^2 \leq 0$, that is $\Sigma_{k=2}^{\infty}(k^2 - 2(1-\beta)|a_k|^2 \leq 1-\beta$.

Corollary 1 *If the function $f(z) = z+a_2z^2+a_3z^3+...$, belongs to the class $SL^*(\beta)$, then*

$$|a_k| \leq \sqrt{\frac{1-\beta}{(k^2 - 2(1-\beta))}}$$

for $k \geq 2$.

Theorem 3 *If the function* $f(z) = \Sigma_{k=1}^{\infty} a_k z^k$ *belongs to class* $SL^*(\beta)$*, then* $|a_2| \leq \frac{1-\beta}{2(\beta+1)}, |a_3| \leq \frac{1-\beta}{2(\beta+2)}$*, and* $|a_4| \leq \frac{1-\beta}{2(\beta+3)}$.

These estimates are sharp.
Proof. If $f(z) = \Sigma_{k=1}^{\infty} a_k z^k$ belongs to class $SL^*(\beta)$, then $(1-\beta)f^2(z) = z^2 f'^2(z) - (1-\beta)f^2(z)w(z)$. where w satisfies $w(0) = 0, |w(z)| < 1$. Let us denote $(zf'(z))^2 = \Sigma_{k=2}^{\infty} A_k z^k, f^2(z) = \Sigma_{k=2}^{\infty} B_k z^k, w(z) = \Sigma_{k=1}^{\infty} C_k z^k$.
Then we have $A_k = \Sigma_{l=1}^{k-1} l(k-l) a_l a_{k-l}, B_k = \Sigma_{l=1}^{k-1} a_l a_{k-l}$ and

$$\Sigma_{k=2}^{\infty}(A_k - (1-\beta)B_k)z^k = (1-\beta)(\Sigma_{k=1}^{\infty} C_k z^k)(\Sigma_{k=2}^{\infty} B_k z^k). \tag{2}$$

Thus we have

$$A_2 = a_1 = 1, \quad A_3 = 4a_2 a_1 = 4a_2, \quad A_4 = 6a_3 + 4a_2^2,$$

and

$$A_5 = 8a_1 a_4 + 12 a_2 a_3, \tag{3}$$

also

$$B_2 = a_1 = 1, \quad B_3 = 2a_2, \quad B_4 = 2a_3 + a_2^2$$

and

$$B_5 = 2a_1 a_4 + 2a_2 a_3. \tag{4}$$

Equating the second and third coefficients of both side of (2) we obtain:

(i) $A_3 - (1-\beta)B_3 = C_1 B_2$

(ii) $A_4 - (1-\beta^2)B_4 = C_1 B_3 + C_2 B_2$

(iii) $A_5 - (1-\beta^2)B_5 = C_1 B_4 + C_2 B_3 + C_3 B_5$.

It is well known that $|C_k| \leq 1$ and $\Sigma_{k=1}^{\infty}|C_k|^2 \leq 1$, therefore we obtain by (3) and (4) that

$$|a_2| \leq \frac{1-\beta}{2(1+\beta)}, \quad |a_3| \leq \frac{1-\beta}{4+2\beta}, \qquad |a_4| \leq \frac{1-\beta}{2(\beta+3)}. \tag{5}$$

Conjecture. Let $f \in SL^*(\beta)$ and $f(z) = \Sigma_{k=1}^{\infty} a_k z^k$.T hen $|a_{n+1}| \leq \frac{1-\beta}{2(\beta+n)}$. This is yet to be proven.

Next, we refer to a classical result of Fekete and Szegö [3] to determine the maximum value of $|a_3 - \mu a_2^2|$ for functions f belonging to H whenever μ is real. Other work related to the functional of Fekete and Szegö can be found in [7].

3. Fekete-Szegö for the class $SL^*(\beta)$

In order to prove our result we have to recall the following lemma:

Lemma 1 *[2] Let h be analytic in $\mathbb{U}$ with $Re\, h(z) > 0$ and be given by $h(z) = 1 + c_1 z + c_2 z^2 + ...$ for $z \in \mathbb{U}$, then*

$$|c_2 - \frac{c_1^2}{2}| \leq 2 - \frac{|c_1|^2}{2}.$$

Theorem 4 *Let f be given by (1) and belongs to the class $SL^*(\beta)$. Then, for $0 \leq \beta < 1$, and*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{1-\beta}{\beta+2} & if \quad \mu \leq \frac{1+3\beta}{2(\beta+2)}, \\ \frac{1-\beta}{2(\beta+2)}(1 - \mu\frac{1-\beta}{2(\beta+2)}) & if \quad \mu \geq \frac{1+3\beta}{2(\beta+2)}. \end{cases}$$

Proof. $a_3 - \mu a_2^2 = \frac{(1-\beta)^2(1+3\beta)}{8(1+\beta)^2(\beta+2)} c_1^2 + \frac{(1-\beta)}{2(2+\beta)} c_2 - \mu \frac{(1-\beta)^2}{4(1+\beta)^2} c_1^2,$

$$= \frac{(1-\beta)}{2(2+\beta)}(c_2 - \frac{c_1^2}{2}) + \frac{1}{2}(\frac{1-\beta}{2(2+\beta)})c_1^2 + \frac{(1-\beta)^2(1+3\beta) - 2\mu(\beta+2)(1-\beta)^2}{8(1+\beta)^2(\beta+2)} c_1^2,$$

$$\begin{aligned} |a_3 - \mu a_2^2| &\leq \frac{(1-\beta)^2(1+3\beta) + 2(1+\beta)^2(1-\beta) - 2\mu(\beta+2)(1-\beta)^2}{8(1+\beta)^2(\beta+2)} |c_1|^2 \\ &+ \frac{(1-\beta)}{2(2+\beta)}(2 - \frac{|c_1|^2}{2}) \\ &= \phi(x), with \quad x = |c_1|, \end{aligned}$$

where we have used Lemma 1. and equations

$$a_2 = \frac{1-\beta}{2(1+\beta)} c_1,$$

and

$$a_3 = \frac{(1-\beta)^2(1+3\beta)}{8(1+\beta)^2(\beta+2)} c_1^2 + \frac{(1-\beta)}{2(2+\beta)} c_2.$$

Elementary calculation indicates that the function attains its maximum value at

$$x_o = 0$$

and thus establishing

$$|a_3 - \mu a_2^2| \leq \phi(x_0) = \frac{1-\beta}{2+\beta}.$$

Next, we have

$$|a_3 - \mu a_2^2| \leq \frac{(1-\beta)^2(1+3\beta) + 2(1+\beta)^2(1-\beta) - 2\mu(\beta+2)(1-\beta)^2}{8(1+\beta)^2(\beta+2)}|c_1|^2$$

$$+\frac{1-\beta}{2+\beta} - \frac{1-\beta}{4(2+\beta)}|c_1|^2$$

$$= \frac{(1-\beta)^2(1+3\beta) - 2\mu(\beta+2)(1-\beta)^2}{8(1+\beta)^2(\beta+2)}|c_1|^2 + \frac{1-\beta}{2+\beta}$$

Secondly, we consider the case $\mu \geq \frac{1+3\beta}{2(\beta+2)}$.

Write

$$a_3 - \mu a_2^2 = a_3 - \frac{1+3\beta}{2(\beta+2)}a_2^2 + (\frac{1+3\beta}{2(\beta+2)} - \mu)a_2^2.$$

From (9), we have

$$|a_2| \leq \frac{1-\beta}{2(\beta+1)},$$

and

$$|a_3| \leq \frac{1-\beta}{2(\beta+2)}.$$

Then

$$|a_3 - \mu a_2^2| \leq |a_3 - \frac{1+3\beta}{2(\beta+2)}a_2^2| + (\frac{1+3\beta}{2(\beta+2)} - \mu)|a_2|^2,$$

and hence $|a_3 - \mu a_2^2| \leq \frac{1-\beta}{2(\beta+2)} - \mu(\frac{1-\beta}{2(\beta+2)})^2$.

The proof of Theorem 3.1 is now complete.

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