Acta Universitatis Apulensis No. 26/2011
ISSN: 1582-5329 pp. 121-128

SOME STABILITY THEOREMS ASSOCIATED WITH $A-$DISTANCE AND $E-$DISTANCE IN UNIFORM SPACES

Alfred Bosede, Gbenga Akinbo

Abstract. In this paper, we establish some stability results for selfmappings in uniform spaces by employing the notion of comparison function as well as the concepts of an $A-$distance and an $E-$distance introduced by Aamri and El Montawakil. Our results improve and unify some of the known stability results in literature.

2000 *Mathematics Subject Classification*: 47H10, 54H25.

1. Introduction

Let X be a nonempty set and let Φ be a nonempty family of subsets of $X \times X$. The pair (X, Φ) is called a uniform space if it satisfies the following properties:
(i) if G is in Φ, then G contains the diagonal $\{(x, x)|x \in X\}$;
(ii) if G is in Φ and H is a subset of $X \times X$ which contains G, then H is in Φ;
(iii) if G and H are in Φ, then $G \bigcap H$ is in Φ;
(iv) if G is in Φ, then there exists H in Φ, such that, whenever (x, y) and (y, z) are in H, then (x, z) is in H;
(v) if G is in Φ, then $\{(y, x)|(x, y) \in G\}$ is also in Φ.
Φ is called the uniform structure of X and its elements are called entourages or neighbourhoods or surroundings.
If property (v) is omitted, then (X, Φ) is called a quasiuniform space. [For examples, see Bourbaki [4] and Zeidler [20]].

Several researchers such as Berinde [3], Jachymski [7], Kada et al [8], Rhoades [13, 14], Rus [16], Wang et al [18] and Zeidler [20] studied the theory of fixed point or common fixed point for contractive selfmappings in complete metric spaces or Banach spaces in general.

Within the last two decades, Kang [9], Rodrı́guez-Montes and Charris [15] established some results on fixed and coincidence points of maps by means of appropriate W-contractive or W-expansive assumptions in uniform space.

2. Preliminaries

Later, Aamri and El Moutawakil [1] proved some common fixed point theorems for some new contractive or expansive maps in uniform spaces by introducing the notions of an A-distance and an E-distance.
Aamri and El Moutawakil [1] introduced and employed the following contractive definition: Let $f, g : X \longrightarrow X$ be selfmappings of X. Then, we have

$$p(f(x), f(y)) \leq \psi(p(g(x), g(y))), \qquad forall \quad x, y \in X, \tag{1}$$

where $\psi : \Re^+ \longrightarrow \Re^+$ is a nondecreasing function satisfying
(i) for each $t \in (0, +\infty), \quad 0 < \psi(t)$,
(ii) $\lim_{n\longrightarrow\infty} \psi^n(t) = 0, \quad \forall\ t \in (0, +\infty)$.
ψ satisfies also the condition $\psi(t) < t$, for each $t > 0, \quad t \in \Re^+$.

In this paper, we shall establish some stability results for selfmappings in uniform spaces by employing the concepts of an A-distance, an E-distance as well as the notion of comparison function.

The following definitions shall be required in the sequel.
Let (X, Φ) be a uniform space. Without loss of generality, $(X, \tau(\Phi))$ denotes a topological space whenever topological concepts are mentioned in the context of a uniform space (X, Φ). [For instance, see Aamri and El Moutawakil [1]]. Definitions $1 - 6$ are contained in Aamri and El Moutawakil [1].

Definition 1. *If $H \in \Phi$ and $(x, y) \in H, (y, x) \in H$, x and y are said to be H-close. A sequence $\{x_n\}_{n=0}^{\infty} \subset X$ is said to be a Cauchy sequence for Φ if for any $H \in \Phi$, there exists $N \geq 1$ such that x_n and x_m are H-close for $n, m \geq N$.*

Definition 2. *A function $p : X \times X \longrightarrow \Re^+$ is said to be an A-distance if for any $H \in \Phi$, there exists $\delta > 0$ such that if $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ for some $z \in X$, then $(x, y) \in H$.*

Definition 3. *A function $p : X \times X \longrightarrow \Re^+$ is said to be an E-distance if*
(p_1) p is an A-distance,
(p_2) $p(x, y) \leq p(x, z) + p(z, y), \quad \forall \ x, y, z \in X$.

Definition 4. *A uniform space (X, Φ) is said to be $Hausdorff$ if and only if the intersection of all $H \in \Phi$ reduces to the diagonal $\{(x, x) | x \in X\}$, i.e. if $(x, y) \in H$ for all $H \in \Phi$ implies $x = y$. This guarantees the uniqueness of limits of sequences. $H \in \Phi$ is said to be symmetrical if $H = H^{-1} = \{(y, x) | (x, y) \in H\}$.*

Definition 5. *Let (X, Φ) be a uniform space and p be an A-distance on X.*
(i) X is said to be S-complete if for every p-Cauchy sequence $\{x_n\}_{n=0}^{\infty}$, there exists $x \in X$ with $\lim_{n\longrightarrow\infty} p(x_n, x) = 0$.

(ii) X is said to be p-Cauchy complete if for every p-Cauchy sequence $\{x_n\}_{n=0}^{\infty}$, there exists $x \in X$ with $\lim_{n \longrightarrow \infty} x_n = x$ with respect to $\tau(\Phi)$.
(iii) $f : X \longrightarrow X$ is said to be p-continuous if $\lim_{n \longrightarrow \infty} p(x_n, x) = 0$ implies that $\lim_{n \longrightarrow \infty} p(f(x_n), f(x)) = 0$.
(iv) $f : X \longrightarrow X$ is $\tau(\Phi)$-continuous if $\lim_{n \longrightarrow \infty} x_n = x$ with respect to $\tau(\Phi)$ implies $\lim_{n \longrightarrow \infty} f(x_n) = f(x)$ with respect to $\tau(\Phi)$.
(v) X is said to be p-bounded if $\delta_p = sup\{p(x, y)|x, y \in X\} < \infty$.

Definition 6. *Let (X, Φ) be a Hausdorff uniform space and p an A-distance on X. Two selfmappings f and g on X are said to be p-compatible if, for each sequence $\{x_n\}_{n=0}^{\infty}$ of X such that $\lim_{n \longrightarrow \infty} p(f(x_n), u) = \lim_{n \longrightarrow \infty} p(g(x_n), u) = 0$ for some $u \in X$, then we have $\lim_{n \longrightarrow \infty} p(f(g(x_n)), g(f(x_n))) = 0$.*

The following definition contained in Berinde [2, 3], Rus [16] and Rus et al [17] shall also be required in the sequel.

Definition 7. *A function $\psi : \Re^+ \longrightarrow \Re^+$ is called a comparison function if*
(i) ψ is monotone increasing;
(ii) $\lim_{n \longrightarrow \infty} \psi^n(t) = 0, \quad \forall\ t \geq 0$.

Many stability results have been obtained within the last decade by various authors using different contractive definitions. Harder and Hicks [5] considered the following concept to obtain various stability results:

Let (X, d) be a complete metric space, $T : X \longrightarrow X$ a selfmap of X. Suppose that $F_T = \{u \in X : Tu = u\}$ is the set of fixed points of T in X.
Let $\{x_n\}_{n=0}^{\infty} \subset X$ be the sequence generated by an iteration procedure involving the operator T, that is,

$$x_{n+1} = h(T, x_n), \quad n = 0, 1, 2, ... \tag{2}$$

where $x_0 \in X$ is the initial approximation and h is some function. Suppose $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point u of T. Let $\{y_n\}_{n=0}^{\infty} \subset X$ and set

$$\epsilon_n = d(y_{n+1}, h(T, y_n)), \quad n = 0, 1, 2, ... \tag{3}$$

Then, the iteration procedure (2) is said to be $T-$***stable*** or stable with respect to T if and only if $\lim_{n \longrightarrow \infty} \epsilon_n = 0$ implies $\lim_{n \longrightarrow \infty} y_n = u$.

Throughout this paper, h represents some function, while f and g shall denote two selfmappings of a uniform space (X, Φ).

We shall employ the following definition of stability of iteration process which is an extension of that of Harder and Hicks [5]:

Definition 8. *Let (X, Φ) be a uniform space and $f, g, : X \longrightarrow X$ two selfmaps of X. Suppose that $F_f \cap F_g \neq \phi$, where $F_f \cap F_g = u$ is the common fixed point of f and g in X; while F_f and F_g are the sets of fixed points of f and g in X respectively.*

Let $\{x_n\}_{n=0}^{\infty} \subset X$ *be the sequence generated by an iteration procedure involving the operators* f *and* g*, that is,*

$$x_{n+1} = h(f, g, x_n), \qquad n = 0, 1, 2, ... \tag{4}$$

where $x_0 \in X$ *is the initial approximation and* h *is some function. Suppose* $\{x_n\}_{n=0}^{\infty}$ *converges to a common fixed point* u *of* f *and* g *in* X*. Let* $\{y_n\}_{n=0}^{\infty} \subset X$ *and set*

$$\epsilon_n = p(y_{n+1}, h(f, g, y_n)), \qquad n = 0, 1, 2, ... \tag{5}$$

where p *is an* $A-$*distance which replaces the distance function* d *in (3).*

Then, the iteration procedure (4) is said to be $(f, g)-$stable or stable with respect to f and g if and only if $\lim_{n\longrightarrow\infty} \epsilon_n = 0$ implies $\lim_{n\longrightarrow\infty} y_n = u$.

Remark 1. If $f = g = T$ in (4), then we obtain the iteration procedure of Harder and Hicks [5]. Also, if $f = g = T$ and $p = d$ in (5), then we get (3); which was used by Harder and Hicks [5] and many other authors.

Our aim in this paper is to establish some stability results for selfmappings in uniform spaces by employing the concepts of an A-distance, an E-distance as well as the notion of comparison function using a more general contractive condition than (1) of Aamri and El Moutawakil [1].

We shall employ the following contractive definition: Let $f, g : X \longrightarrow X$ be selfmappings of a uniform space X. There exist $L \geq 0$ and $\psi : \Re^+ \longrightarrow \Re^+$ a comparison function (or just a continuous monotone increasing function with conditions (i) and (ii) of inequality (1)) satisfying

$$p(f(x), f(y)) \leq e^{Lp(x,g(x))}\psi(p(g(x), g(y))), \qquad \forall x, y \in X. \tag{6}$$

where p is an $A-$distance in X.

Remark 2. The contractive condition (6) is more general than (1) in the sense that if $L = 0$ in the above inequality, then we obtain (1), which was employed by Aamri and El Moutawakil [1].

3.Main results

Theorem 1.*Let* (X, Φ) *be a Hausdorff uniform space and* p *an* A*-distance on* X *such that* X *is* p*-bounded and* S*-complete. For arbitrary* $x_0 \in X$*, define a sequence* $\{x_n\}_{n=0}^{\infty}$ *iteratively by*

$$x_{n+1} = f(x_n), \qquad n = 0, 1, 2, ... \tag{7}$$

Suppose that f and g are commuting p-continuous or $\tau(\Phi)$-continuous selfmappings of X, with a common fixed point u in X, satisfying
(i) $f(X) \subseteq g(X)$;
(ii) $p(f(x_i), f(x_i)) = 0, \quad \forall\, x_i \in X, \quad i = 0, 1, 2, ...$ In particular, $p(f(u), f(u)) = 0$;
(iii) $f, g : X \longrightarrow X$ satisfy the contractive condition (6).
Suppose also that $\psi : \Re^+ \longrightarrow \Re^+$ is a comparison function (or just a continuous monotone increasing function with conditions (i) and (ii) of inequality (1)).
Then, iteration (7) is $(f, g)-$stable.

Proof. For arbitrary $x_0 \in X$, select $x_1 \in X$ such that $f(x_0) = g(x_1)$. Similarly, for $x_1 \in X$, select $x_2 \in X$ such that $f(x_1) = g(x_2)$. Continuing this process, we select $x_n \in X$ such that $f(x_{n-1}) = g(x_n)$. Hence, iteration (7) is well-defined.
Let $\{y_n\}_{n=0}^{\infty} \subset X$ and let $\{\epsilon_n\}_{n=0}^{\infty}$ be a sequence defined by $\epsilon_n = p(y_{n+1}, f(y_n))$.

Suppose that $\{x_n\}_{n=0}^{\infty}$ converges to a common fixed point u of f and g in X. Suppose also that $\lim_{n\longrightarrow\infty} \epsilon_n = 0$. Then, we shall prove that $\lim_{n\longrightarrow\infty} y_n = u$.

Since X is $p-$bounded, we assume that $p(f(u), f(y_0)) \leq \delta_p(X)$, $y_0 \in X$, where $\delta_p(X) = \sup\{p(x, y)|x, y \in X\} < +\infty$.

Indeed, since $x_n = f(x_{n-1}),\ n = 1, 2, ...,$ then, using the contractive definition (6) and the triangle inequality, we obtain

$$\begin{aligned}
p(y_{n+1}, u) &\leq p(y_{n+1}, f(y_n)) + p(f(y_n), u) \\
&= \epsilon_n + p(f(y_n), f(u)) \\
&= \epsilon_n + p(f(u), f(y_n)) \\
&\leq \epsilon_n + e^{Lp(u,g(u))}\psi(p(g(u), g(y_n))) \\
&= \epsilon_n + e^{Lp(f(u),f(u))}\psi(p(f(u), f(y_{n-1}))) \\
&= \epsilon_n + e^{L(0)}\psi(p(f(u), f(y_{n-1}))) \\
&= \epsilon_n + e^{0}\psi(p(f(u), f(y_{n-1}))) \\
&= \epsilon_n + \psi(p(f(u), f(y_{n-1}))) \\
&\leq \epsilon_n + \psi[e^{Lp(u,g(u))}\psi(p(g(u), g(y_{n-1})))] \\
&= \epsilon_n + \psi[e^{Lp(f(u),f(u))}\psi(p(f(u), f(y_{n-2})))] \\
&= \epsilon_n + \psi[e^{L(0)}\psi(p(f(u), f(y_{n-2})))] \\
&= \epsilon_n + \psi[e^{0}\psi(p(f(u), f(y_{n-2})))] \\
&= \epsilon_n + \psi^2(p(f(u), f(y_{n-2}))) \\
&\leq ... \leq \epsilon_n + \psi^n(p(f(u), f(y_0))) \\
&\leq \epsilon_n + \psi^n(\delta_p(X)). \qquad (8)
\end{aligned}$$

But condition (ii) of Definition 7 of a comparison function gives

$$\lim_{n\longrightarrow\infty} \psi^n(\delta_p(X)) = 0.$$

Hence, taking the limit as $n \longrightarrow \infty$ of both sides of (8) yields
$p(y_{n+1}, u) \longrightarrow 0, \quad as \quad n \longrightarrow \infty$, which implies that $\lim_{n\longrightarrow\infty} y_n = u$.

Conversely, let $\lim_{n\longrightarrow\infty} y_n = u$. Then,

$$\begin{aligned}
\epsilon_n &= p(y_{n+1}, f(y_n)) \\
&\leq p(y_{n+1}, u) + p(u, f(y_n)) \\
&= p(y_{n+1}, u) + p(fu, f(y_n)) \\
&\leq p(y_{n+1}, u) + e^{Lp(u,g(u))}\psi(p(g(u), g(y_n))) \\
&= p(y_{n+1}, u) + e^{Lp(f(u),f(u))}\psi(p(f(u), f(y_{n-1}))) \\
&= p(y_{n+1}, u) + e^{L(0)}\psi(p(f(u), f(y_{n-1}))) \\
&= p(y_{n+1}, u) + e^{0}\psi(p(f(u), f(y_{n-1}))) \\
&= p(y_{n+1}, u) + \psi(p(f(u), f(y_{n-1}))) \\
&\leq p(y_{n+1}, u) + \psi[e^{Lp(u,g(u))}\psi(p(g(u), g(y_{n-1})))] \\
&= p(y_{n+1}, u) + \psi[e^{Lp(f(u),f(u))}\psi(p(f(u), f(y_{n-2})))] \\
&= p(y_{n+1}, u) + \psi[e^{L(0)}\psi(p(f(u), f(y_{n-2})))] \\
&= p(y_{n+1}, u) + \psi[e^{0}\psi(p(f(u), f(y_{n-2})))] \\
&= p(y_{n+1}, u) + \psi^2(p(f(u), f(y_{n-2}))) \\
&\leq \ldots \leq p(y_{n+1}, u) + \psi^n(p(f(u), f(y_0))) \\
&\leq p(y_{n+1}, u) + \psi^n(\delta_p(X)) \longrightarrow 0 \quad as \quad n \longrightarrow \infty
\end{aligned}$$

This completes the proof.

The next theorem is where p is an E-distance on X.

Theorem 2. *Let $(X, \Phi), f, g, u, \psi, \{x_n\}_{n=0}^{\infty}$ be as defined in Theorem 1 above and p an E-distance on X.*
Then, iteration (7) is $(f, g)-$stable.

Proof. We observe that an E-distance function p on X is also an A-distance on X.
Therefore, the remaining part of the proof follows the same standard method as in the proof of Theorem 1 above and it is therefore omitted.

Remark 3. Theorem 1 and Theorem 2 of this paper are generalizations of those of Berinde [3], Harder and Hicks [5] and many others; and this is also a further improvement to many existing stability results in literature.

References

[1] M. Aamri, D. El Moutawakil, *Common fixed point theorems for E-contractive or E-expansive maps in uniform spaces*, Acta Math. Acad. Paedagog. Nyházi. (N.S.), 20, 1, (2004), 83-91.

[2] V. Berinde, *Iterative approximation of fixed points*, Editura Efemeride, Baia Mare, 2002.

[3] V. Berinde, *On stability of some fixed point procedures*, Bul. Śtiinť. Univ. Baia Mare, Ser. B, Mathematică-Informatică, 18, 1, (2002), 7-14.

[4] N. Bourbaki, *Éléments de mathématique. Fasc. II. Livre III: Topologie générale*, Chapitre 1: Structures topologiques. Chapitre 2: Structures uniformes. Quatriéme édition. Actualités Scientifiques et Industrielles, no. 1142. Hermann, Paris, 1965.

[5] A. M. Harder, T. L. Hicks, *Stability results for fixed point iteration procedures*, Math. Japonica, 33, 5, (1988), 693-706.

[6] C. O. Imoru, M. O. Olatinwo, *On the stability of Picard and Mann iteration processes*, Carpathian J. Math., 19, 2, (2003), 155-160.

[7] J. Jachymski, *Fixed point theorems for expansive mappings*, Math. Japonica, 42, 1, (1995), 131-136.

[8] O. Kada, T. Suzuki, W. Takahashi, *Nonconvex minimization theorems and fixed point theorems in complete metric spaces*, Math. Japonica, 44, 2, (1996), 381-391.

[9] S. M. Kang, *Fixed points for expansion mappings*, Math. Japonica, 38, 4, (1993), 713-717.

[10] M. O. Olatinwo, *Some common fixed point theorems for selfmappings in uniform spaces*, Acta Math. Acad. Paedagog. Nyházi. (N.S.), 23, (2007), 47-54.

[11] M. O. Osilike, *Some stability results for fixed point iteration procedures*, J. Nigerian Math. Soc., 14/15, (1995), 17-29.

[12] B. E. Rhoades, *A comparison of various definitions of contractive mappings*, Trans. Amer. Math. Soc., 226, (1977), 257-290.

[13] B. E. Rhoades, *Fixed point theorems and stability results for fixed point iteration procedures*, Indian J. Pure Appl. Math., 21, 1, (1990), 1-9.

[14] B. E. Rhoades, *Fixed point theorems and stability results for fixed point iteration procedures II*, Indian J. Pure Appl. Math., 24, 11, (1993), 691-703.

[15] J. Rodrígues-Montes, J. A. Charris, *Fixed points for W-contractive or W-expansive maps in uniform spaces: toward a unified approach*, Southwest J. Pure Appl. Math., 1, (2001), 93-101.

[16] I. A. Rus, *Generalized Contractions and Applications*, Cluj University Press, Cluj-Napoca, 2001.

[17] I. A. Rus, A. Petrusel, G. Petrusel, *Fixed Point Theory: 1950-2000*, Romanian Contributions, House of the Book of Science, Cluj-Napoca, 2002.

[18] S. Z. Wang, B. Y. Li, Z. M. Gao, K. Iséki, *Some fixed point theorems on expansion mappings*, Math. Japonica, 29, 4, (1984), 631-636.

[19] T. Zamfirescu, *Fix point theorems in metric spaces*, Arch. Math., 23, (1972),

292-298.

[20] E. Zeidler, *Nonlinear Functional Analysis and its Applications I*, Springer-Verlag, New York, 1986.

Alfred Olufemi Bosede
Department of Mathematics
Lagos State University
Ojoo, Nigeria.
email: *aolubosede@yahoo.co.uk*

Gbenga Akinbo
Department of Mathematics
Obafemi Awolowo University
Ile-Ife, Nigeria.
email: *agnebg@yahoo.co.uk*