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SUBORDINATION RESULTS FOR A CLASS OF NON-BAZILEVIČ FUNCTIONS WITH RESPECT TO SYMMETRIC POINTS

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Abstract. In this article, we investigate a new class of non-Bazilevič functions with respect to k-symmetric points defined by a generalized differential operator. Several interesting subordination results are derived for the functions belonging to this class in the open unit disk.

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1. Introduction and preliminaries

Let $\mathcal{H}(a,n)$ denote the class of functions $f(z)$ of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots , \quad (z \in \mathcal{U}), \tag{1}$$

which are analytic in the unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. In particular, let $\mathcal{A}$ be the subclass of $\mathcal{H}(0,1)$ containing functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{2}$$

We denote by S, S^*, K and C, the classes of all functions in $\mathcal{A}$ which are, respectively, univalent, starlike, convex and close-to-convex in $\mathcal{U}$. Let $f(z)$ and $g(z)$ be analytic in $\mathcal{U}$. Then we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathcal{U}$, if there exists an analytic function $w(z)$ in $\mathcal{U}$ with $w(0) = 0, \quad |w(z)| < 1 \quad (z \in \mathcal{U})$, such that $f(z) = g(w(z)) \qquad (z \in \mathcal{U})$.

We denote this subordination by $f(z) \prec g(z)$. Furthermore, if the function $g(z)$ is univalent in $\mathcal{U}$, then $f(z) \prec g(z) \quad (z \in \mathcal{U}) \Longleftrightarrow f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$.

Let k be a positive integer and let $\varepsilon_k = \exp(\frac{2\pi i}{k})$. For $f \in \mathcal{A}$ let

$$f_k(z) = \frac{1}{k}\sum_{j=0}^{k-1} \varepsilon_k^{-j} f(\varepsilon_k^j z). \tag{3}$$

The function f is said to be starlike with respect to k-symmetric points if it satisfies

$$Re\left(\frac{zf'(z)}{f_k(z)}\right) > 0, \quad z \in \mathcal{U}. \tag{4}$$

We denote by $S_s^{(k)}$ the subclass of $\mathcal{A}$ consisting of all functions starlike with respect to k-symmetric points in $\mathcal{U}$. The class $S_s^{(2)}$ was introduced and studied by K. Sakaguchi [8]. If j is an integer, then the following identities follow directly from (3).

$$\begin{aligned} f_k(\varepsilon^j z) &= \varepsilon^j f_k(z),\\ f_k'(\varepsilon^j z) &= f_k'(z) = \frac{1}{k}\sum_{j=0}^{k-1} f'(\varepsilon_k^j z),\\ \varepsilon^j f_k''(\varepsilon^j z) &= f_k''(z) = \frac{1}{k}\sum_{j=0}^{k-1} \varepsilon^j f''(\varepsilon_k^j z). \end{aligned} \tag{5}$$

If we replace z by $\varepsilon^j z$ in (4) and take the sum with respect to j from 0 to $k-1$, then we obtain

$$Re\left(\frac{zf_k'(z)}{f_k(z)}\right) > 0, \quad z \in \mathcal{U}.$$

This shows that if $f \in S_s^{(k)}$, then $f_k \in S^*$. Using this together with the condition (4) we see that functions in $S_s^{(k)}$ are close-to-convex. We also note that different subclasses of $S_s^{(k)}$ can be obtained by replacing condition (4) by

$$Re\left(\frac{zf'(z)}{f_k(z)}\right) \prec h(z),$$

where $h(z)$ is a given convex function, with $h(0) = 1$ and $Re\, h(z) > 0$.
We will make use of the following definition of fractional derivatives by S. Owa [6]. The fractional derivative of order δ is defined, for a function f, by

$$D_z^\delta f(z) = \frac{1}{\Gamma(1-\delta)}\frac{d}{dz}\int_0^z \frac{f(\xi)}{(z-\xi)^\delta}\, d\xi, \quad (0 \le \delta < 1) \tag{6}$$

where the function f is analytic in a simply connected region of the complex z-plane containing the origin, and the multiplicity of $(z-\xi)^{-\delta}$ is removed by requiring $log(z-\xi)$ to be real when $(z-\xi)>0$. It follows from (6) that

$$D_z^{\delta} z^n = \frac{\Gamma(n+1)}{\Gamma(n+1-\delta)} z^{n-\delta} \quad (0 \leq \delta < 1,\, n \in \mathbb{N} = \{1, 2, \ldots\}).$$

Using $D_z^{\delta} f,$ S. Owa and H. M. Srivastava [7] introduced the operator $\Omega^{\delta} : \mathcal{A} \longrightarrow \mathcal{A}$, which is known as an extension of fractional derivative and fractional integral as follows: $\Omega^{\delta} f(z) = \Gamma(2-\delta)\, z^{\delta}\, D_z^{\delta}\, f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n+1-\delta)}\, a_n z^n$. Here we note that $\Omega^0 f(z) = f(z)$.
In [2] F. M. Al-Oboudi and K. A. Al-Amoudi defined the linear multiplier fractional differential operator $D_{\lambda}^{m,\delta}$ as follows:

$$\begin{aligned}
D_{\lambda}^{0,0} f(z) &= f(z), \\
D_{\lambda}^{1,\delta} f(z) &= (1-\lambda)\Omega^{\delta} f(z) + \lambda\, z(\Omega^{\delta} f(z))' \\
&= D_{\lambda}^{\delta}(f(z)), \quad (0 \leq \delta < 0,\ \lambda \geq 0), \\
D_{\lambda}^{2,\delta} f(z) &= D_{\lambda}^{\delta}(D_{\lambda}^{1,\delta} f(z)), \\
&\ \vdots \\
D_{\lambda}^{m,\delta} f(z) &= D_{\lambda}^{1,\delta}(D_{\lambda}^{m-1,\delta} f(z)), \qquad m \in \mathbb{N}.
\end{aligned} \tag{7}$$

If $f(z)$ is given by (2), then by (7), we have

$$D_{\lambda}^{m,\delta} f(z) = z + \sum_{n=2}^{\infty} \left(\frac{\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n+1-\delta)} \big[1 + (n-1)\lambda\big] \right)^m a_n z^n.$$

It can be seen that, by specializing the parameters the operator $D_{\lambda}^{m,\delta} f(z)$ reduces to many known and new integral and differential operators. In particular, when $\delta = 0$ the operator $D_{\lambda}^{m,\delta}$ reduces to the operator introduced by F. AL-Oboudi [1] and for $\delta = 0$, $\lambda = 1$ it reduces to the operator introduced by G. S. Sălăgean [9]. Further we remark that, when $m = 1$, $\lambda = 0$ the operator $D_{\lambda}^{m,\delta} f(z)$ reduces to Owa-Srivastava fractional differential operator [7].
Throughout this paper, we assume that

$$f_k^m(\lambda, \delta; z) = \frac{1}{k} \sum_{j=0}^{k-1} \varepsilon_k^{-j} (D_{\lambda}^{m,\delta} f(\varepsilon_k^j z)) = z + \cdots, \qquad (f \in \mathcal{A}).$$

Clearly, for $k=1$, we have $f_1^m(\lambda,\delta;z)=D_\lambda^{m,\delta}f(z)$. Let $\mathcal{P}$ denote the class of analytic functions $h(z)$ with $h(0)=1$, which are convex and univalent in $\mathcal{U}$ and for which $Re\{h(z)\}>0, \quad (z\in\mathcal{U})$.

We now introduce the following subclass of $\mathcal{A}$:

Definition 1.1. *A function $f\in\mathcal{A}$ is said to be in the class $\mathcal{N}_k^m(\lambda,\delta,\gamma;\phi)$ if and only if*

$$\left(D_\lambda^{m,\delta}f(z)\right)'\left(\frac{z}{f_k^m(\lambda,\delta;z)}\right)^{1+\gamma}\prec\phi(z), \qquad (z\in\mathcal{U}). \tag{8}$$

where $0\le\gamma\le1$, $\phi\in\mathcal{P}$ and $f_k^m(\lambda,\delta;z)\neq0$ for all $z\in\mathcal{U}\setminus\{0\}$.

We remark that for the choice of $\phi(z)=\dfrac{1+z}{1-z}$, $m=0$, $k=1$ the class $\mathcal{N}_k^m(\lambda,\delta,\gamma;\phi)$ reduces to $\mathcal{N}(\gamma)$, $(0<\gamma<1)$ introduced by Obradović in [5]. He named this class of functions as non-Bazilevič type.

In this paper, we derive some sufficient conditions for functions belonging to the class $\mathcal{N}_k^m(\lambda,\delta,\gamma;\phi)$. In order to prove our results we need the following lemmas.

Lemma 1.2. [10] *Let h be convex in $\mathcal{U}$, with $h(0)=a, \delta\neq0$ and $Re\,\delta\ge0$. If $p\in\mathcal{H}(a,n)$ and*

$$p(z)+\frac{zp'(z)}{\delta}\prec h(z),$$

then

$$p(z)\prec q(z)\prec h(z),$$

where

$$q(z)=\frac{\delta}{n\,z^{\delta/n}}\int_0^z h(t)\,t^{(\delta/n)-1}\,dt.$$

The function q is convex and is the best (a, n)-dominant.

Lemma 1.3. [3] *Let h be starlike in $\mathcal{U}$, with $h(0)=0$. If $p\in\mathcal{H}(a,n)$ satisfies*

$$zp'(z)\prec h(z),$$

then

$$p(z)\prec q(z)=a+n^{-1}\int_0^z h(t)\,t^{-1}\,dt.$$

The function q is convex and is the best (a, n)-dominant.

Lemma 1.4. [4] *Let $q(z)$ be univalent in the unit disc $\mathcal{U}$ and let $\theta(z)$ be analytic in a domain D containing $q(\mathcal{U})$. If $zq'(z)\theta(q(z))$ is starlike in $\mathcal{U}$ and*

$$zp'(z)\theta(p(z))\prec zq'(z)\theta(q(z))$$

then $p(z)\prec q(z)$ and $q(z)$ is the best dominant.

2. Main results

Theorem 2.1. *Let $f \in \mathcal{A}$ with $f(z)$ and $f'(z) \neq 0$ for all $z \in \mathcal{U}\backslash\{0\}$ and let h be convex in $\mathcal{U}$, with $h(0) = 1$ and $Re\, h(z) > 0$. If*

$$\left\{ \left(D_\lambda^{m,\delta} f(z)\right)' \left(\frac{z}{f_k^m(\lambda,\delta;z)} \right)^{1+\gamma} \right\}^2 \left[3 + 2\gamma + \frac{2z\left(D_\lambda^{m,\delta} f(z)\right)''}{\left(D_\lambda^{m,\delta} f(z)\right)'} - 2(1+\gamma) \frac{z\left(f_k^m(\lambda,\delta;z)\right)'}{f_k^m(\lambda,\delta;z)} \right] \prec h(z), \tag{9}$$

then

$$\left(D_\lambda^{m,\delta} f(z)\right)' \left(\frac{z}{f_k^m(\lambda,\delta;z)} \right)^{1+\gamma} \prec q(z) = \sqrt{Q(z)}, \tag{10}$$

where

$$Q(z) = \frac{1}{z} \int_0^z h(t)dt,$$

and the function q is the best dominant.

Proof. Let $p(z) = \left(D_\lambda^{m,\delta} f(z)\right)' \left(\frac{z}{f_k^m(\lambda,\delta;z)} \right)^{1+\gamma} \qquad (z \in \mathcal{U}; z \neq 0; f \in \mathcal{A}).$
Then $p(z) \in \mathcal{H}(1,1)$ with $p(z) \neq 0$ in $\mathcal{U}$. Since h is convex, it can be easily verified that Q is convex and univalent. We now set $P(z) = p^2(z)$. Then $P(z) \in \mathcal{H}(1,1)$ with $P(z) \neq 0$ in $\mathcal{U}$. By logarithmic differentiation we have,

$$\frac{zP'(z)}{P(z)} = 2\left[\frac{z\left(D_\lambda^{m,\delta} f(z)\right)''}{\left(D_\lambda^{m,\delta} f(z)\right)'} + (1+\gamma)\left(1 - \frac{z\left(f_k^m(\lambda,\delta;z)\right)'}{f_k^m(\lambda,\delta;z)} \right) \right].$$

Therefore, by (9) we have

$$P(z) + zP'(z) \prec h(z). \tag{11}$$

Now, by Lemma 1.2 with $\delta = 1$, we deduce that

$$P(z) \prec Q(z) \prec h(z),$$

and Q is the best dominant of (11). Since $Re\, h(z) > 0$ and $Q(z) \prec h(z)$ we also have $Re\, Q(z) > 0$. Hence, the univalence of Q implies the univalence of $q(z) = \sqrt{Q(z)}$, and

$$p^2(z) = P(z) \prec Q(z) = q^2(z),$$

which implies that $p(z) \prec q(z)$. Since Q is the best dominant of (11), we deduce that q is the best dominant of (10). This completes the proof.

Corollary 2.2. *Let $f \in \mathcal{A}$ with $f(z)$ and $f'(z) \neq 0$ for all $z \in \mathcal{U}\backslash\{0\}$. If $Re\big(\Psi(z)\big) > \alpha, \quad (0 \leq \alpha < 1)$, where*

$$\Psi(z) = \left\{ \big(D_\lambda^{m,\delta} f(z)\big)' \left(\frac{z}{f_k^m(\lambda,\delta;z)} \right)^{1+\gamma} \right\}^2 \left[3+2\gamma+\frac{2z\big(D_\lambda^{m,\delta} f(z)\big)''}{\big(D_\lambda^{m,\delta} f(z)\big)'} - 2(1+\gamma)\frac{z\big(f_k^m(\lambda,\delta;z)\big)'}{f_k^m(\lambda,\delta;z)} \right],$$

then

$$Re\left\{ \big(D_\lambda^{m,\delta} f(z)\big)' \left(\frac{z}{f_k^m(\lambda,\delta;z)} \right)^{1+\gamma} \right\} > \mu(\alpha),$$

where $\mu(\alpha) = \big[2(1-\alpha).\log 2 + (2\alpha - 1)\big]^{\frac{1}{2}}$, and this result is sharp.

Proof. Let $h(z) = \dfrac{1+(2\alpha-1)z}{1+z}$ with $0 \leq \alpha < 1$. Then from Theorem 2.1, it follows that $Q(z)$ ia convex and $Re\, Q(z) > 0$. Also we have,

$$\min_{|z|\leq 1} Re\, q(z) = \min_{|z|\leq 1} Re\, \sqrt{Q(z)} = \sqrt{Q(1)} = \big[2(1-\alpha).\log 2 + (2\alpha-1)\big]^{\frac{1}{2}}.$$

This completes the proof the corollary.

By setting $m = 0$, and $\gamma = 0$ in Corollary 2.2, we have the following corollary.

Corollary 2.3. *Let $f \in \mathcal{A}$ with $f(z)$ and $f'(z) \neq 0$ for all $z \in \mathcal{U}\backslash\{0\}$. If $Re\left\{ \left(\frac{zf'(z)}{f_k(z)}\right)^2 \left[3+2\frac{zf''(z)}{f'(z)} - 2\frac{zf_k'(z)}{f_k(z)}\right] \right\} > \alpha, \quad (0 \leq \alpha < 1)$, then $Re\left\{\frac{zf'(z)}{f_k(z)}\right\} > \mu(\alpha)$, where $\mu(\alpha) = \big[2(1-\alpha).\log 2 + (2\alpha-1)\big]^{\frac{1}{2}}$, and this result is sharp.*

Theorem 2.4. *Let $f \in \mathcal{A}$ with $f(z)$ and $f'(z) \neq 0$ for all $z \in \mathcal{U}\backslash\{0\}$ and h be starlike in $\mathcal{U}$, with $h(0) = 0$. If*

$$\frac{z\big(D_\lambda^{m,\delta} f(z)\big)''}{\big(D_\lambda^{m,\delta} f(z)\big)'} + (1+\gamma)\left(1 - \frac{z\big(f_k^m(\lambda,\delta;z)\big)'}{f_k^m(\lambda,\delta;z)}\right) \prec h(z) \qquad (z \in \mathcal{U};\ \gamma \geq 0), \tag{12}$$

then

$$\big(D_\lambda^{m,\delta} f(z)\big)' \left(\frac{z}{f_k^m(\lambda,\delta;z)} \right)^{1+\gamma} \prec q(z) = \exp\left(\int_0^z \frac{h(t)}{t}\, dt \right), \tag{13}$$

and q is the best dominant.

Proof. Let $p(z) = \big(D_\lambda^{m,\delta} f(z)\big)' \left(\dfrac{z}{f_k^m(\lambda,\delta;z)} \right)^{1+\gamma} \qquad (z \in \mathcal{U}; z \neq 0; f \in \mathcal{A}).$

Then $p(z) \in \mathcal{H}(1,1)$ with $p(z) \neq 0$ in $\mathcal{U}$. Thus we can define an analytic function $P(z) = \log p(z)$. Clearly $P \in \mathcal{H}(0,1)$, and by (12) we obtain

$$zP'(z) \prec h(z). \tag{14}$$

Now by using Lemma 1.3 we deduce that $P(z) \prec Q(z) = \int_0^z \frac{h(t)}{t}\,dt$, and Q is the best dominant of (14). Converting back we obtain $p(z) = \exp P(z) \prec \exp Q(z) = q(z)$, and since Q is the best dominant of (14), we deduce that q is the best dominant of (13). This completes the proof.

By setting $m = 0$ in Theorem 2.4, we have the following corollary.

Corollary 2.5. *Let $f \in \mathcal{A}$ with $f(z)$ and $f'(z) \neq 0$ for all $z \in \mathcal{U}\backslash\{0\}$ and h be starlike in $\mathcal{U}$, with $h(0) = 0$. If*

$$\frac{zf''(z)}{f'(z)} + (1+\gamma)\left(1 - \frac{zf_k'(z)}{f_k(z)}\right) \prec h(z) \qquad (z \in \mathcal{U};\ \gamma \geq 0),$$

then $f'(z)\left(\frac{z}{f_k(z)}\right)^{1+\gamma} \prec q(z) = \exp\left(\int_0^z \frac{h(t)}{t}\,dt\right)$, *and q is the best dominant.*

Theorem 2.6. *Let $f \in \mathcal{A}$ with $f(z)$ and $f'(z) \neq 0$ for all $z \in \mathcal{U}\backslash\{0\}$ and $q(z)$ be univalent in the unit disc $\mathcal{U}$ with $q'(z) \neq 0$ in $\mathcal{U}$. If $\frac{zq'(z)}{q(z)}$ is starlike in $\mathcal{U}$ and*

$$\frac{z\big(D_\lambda^{m,\delta} f(z)\big)''}{\big(D_\lambda^{m,\delta} f(z)\big)'} + (1+\gamma)\left(1 - \frac{z\big(f_k^m(\lambda,\delta;z)\big)'}{f_k^m(\lambda,\delta;z)}\right) \prec \frac{z\,q'(z)}{q(z)} \qquad (z \in \mathcal{U};\ \gamma \geq 0), \quad (15)$$

then $\big(D_\lambda^{m,\delta} f(z)\big)'\left(\frac{z}{f_k^m(\lambda,\delta;z)}\right)^{1+\gamma} \prec q(z)$, *and $q(z)$ is the best dominant.*

Proof. Let $p(z) = \big(D_\lambda^{m,\delta} f(z)\big)'\left(\dfrac{z}{f_k^m(\lambda,\delta;z)}\right)^{1+\gamma} \qquad (z \in \mathcal{U}; z \neq 0; f \in \mathcal{A}).$

By setting $\theta(\omega) = \frac{a}{\omega}$, $\quad a \neq 0$, it can be easily verified that $\theta(\omega)$ is analytic in $\mathbb{C} - \{0\}$. Then we obtain $a\frac{z\,p'(z)}{p(z)} = a\left[\frac{z\big(D_\lambda^{m,\delta} f(z)\big)''}{\big(D_\lambda^{m,\delta} f(z)\big)'} + (1+\gamma)\left(1 - \frac{z\big(f_k^m(\lambda,\delta;z)\big)'}{f_k^m(\lambda,\delta;z)}\right)\right] \prec a\,\frac{z\,q'(z)}{q(z)}$. Now, the assertion of the theorem follows by an application of Lemma 1.4.

By setting $m = 0$ in Theorem 2.6, we have the following corollary.

Corollary 2.7. *Let $f \in \mathcal{A}$ with $f(z)$ and $f'(z) \neq 0$ for all $z \in \mathcal{U}\backslash\{0\}$ and $q(z)$ be univalent in the unit disc $\mathcal{U}$ with $q'(z) \neq 0$ in $\mathcal{U}$. If $\frac{zq'(z)}{q(z)}$ is starlike in $\mathcal{U}$ and*

$$\frac{zf''(z)}{f'(z)} + (1+\gamma)\left(1 - \frac{zf_k'(z)}{f_k(z)}\right) \prec \frac{zq'(z)}{q(z)} \qquad (z \in \mathcal{U};\ \gamma \geq 0),$$

then $f'(z)\left(\frac{z}{f_k(z)}\right)^{1+\gamma} \prec q(z)$, *and $q(z)$ is the best dominant.*

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