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AN APPLICATION OF SALAGEAN DERIVATIVE ON PARTIAL SUMS OF CERTAIN ANALYTIC AND UNIVALENT FUNCTIONS

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Abstract. Let $\phi(z)$ be a fixed analytic and univalent function of the form $\phi(z) = z + \sum_{k=2}^{\infty} c_k z^k$ and $H_\phi(c_k, \delta)$ be the subclass consisting of analytic and univalent functions f of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ which satisfy the inequality $\sum_{k=2}^{\infty} c_k |a_k| \leq \delta$. In this paper, we determine the sharp lower bounds for $Re\left\{\frac{D^p f(z)}{D^p f_n(z)}\right\}$ and $Re\left\{\frac{D^p f_n(z)}{D^p f(z)}\right\}$, where $f_n(z) = z + \sum_{k=2}^{n} a_k z^k$ be the sequence of partial sums of a function $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ belonging to the class $H_\phi(c_k, \delta)$ and D^p stands for the Salagean derivative. In this paper, we extend the results of ([1], [2], [3], [6]) and we point out that some conditions on the results of Frasin (([1],Theorem 2), ([2],Theorem 2.7)) are incorrect and we correct them.

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1. Introduction

Let A denote the class of functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic in the open unit disc $U = \{z : |z| < 1\}$. Further, by S we shall denote the class of all functions in A which are univalent in U. A function $f(z)$ in S is said to be starlike of order α $(0 \leq \alpha < 1)$, denoted by $S^*(\alpha)$, if it satisfies $Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, (z \in U)$, and is said to be convex of order α $(0 \leq \alpha < 1)$, denoted by $K(\alpha)$, if it satisfies $Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \quad (z \in U)$.

Let $T^*(\alpha)$ and $C(\alpha)$ be subclasses of $S^*(\alpha)$ and $K(\alpha)$, respectively, whose functions are of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, a_k \geq 0. \tag{2}$$

A sufficient condition for a function of the form (1) to be in $S^*(\alpha)$ is that

$$\sum_{k=2}^{\infty} (k-\alpha)|a_k| \leq 1-\alpha \tag{3}$$

and to be in $K(\alpha)$ is that

$$\sum_{k=2}^{\infty} k(k-\alpha)|a_k| \leq 1-\alpha. \tag{4}$$

For the functions of the form (2), Silverman [5] proved that the above sufficient conditions are also necessary.

Let $\phi(z) \in S$ be a fixed function of the form

$$\phi(z) = z + \sum_{k=2}^{\infty} c_k z^k, (c_k \geq c_2 > 0, k \geq 2). \tag{5}$$

Very recently, Frasin [2] defined the class $H_\phi(c_k, \delta)$ consisting of functions $f(z)$ of the form (1) which satisfy the inequality

$$\sum_{k=2}^{\infty} c_k |a_k| \leq \delta, \tag{6}$$

where $\delta > 0$. He shows that for suitable choices of c_k and δ, $H_\phi(c_k, \delta)$ reduces to various known subclasses of S studied by various authors (for detailed study see [2] and references therein).

In the present paper, we determine sharp lower bounds for $Re\left\{\frac{D^p f(z)}{D^p f_n(z)}\right\}$ and $Re\left\{\frac{D^p f_n(z)}{D^p f(z)}\right\}$, where $f_n(z) = z + \sum_{k=2}^{n} a_k z^k$ be the sequence of partial sums of a function $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ belonging to the class $H_\phi(c_k, \delta)$ and the operator D^p stands for the Salagean derivative introduced by Salagean in [4]. In this paper, we extend the results of Frasin ([1], [2]), Rosy et al. [3] and Silverman [6]. Further, we point out that some condition on the results of Frasin ([[1], Theorem 2], [[2], Theorem 2.7]) are incorrect and we correct them.

2. Main Results

Theorem 2.1. *If $f \in H_\phi(c_k, \delta)$, then*

$$(i) \quad Re\left\{\frac{D^p f(z)}{D^p f_n(z)}\right\} \geq \frac{c_{n+1} - (n+1)^p \delta}{c_{n+1}}, \qquad (z \in U) \tag{7}$$

and

$$(ii) \quad Re\left\{\frac{D^p f_n(z)}{D^p f(z)}\right\} \geq \frac{c_{n+1}}{c_{n+1} + (n+1)^p \delta}, \qquad (z \in U) \tag{8}$$

where $c_k \geq \begin{cases} k^p \delta & if \quad k = 2, 3, ..., n \\ \dfrac{k^p c_{n+1}}{(n+1)^p} & if \quad k = n+1, n+2... \end{cases}.$

The results (7) and (8) are sharp with the function given by

$$f(z) = z + \frac{\delta}{c_{n+1}} z^{n+1}, \tag{9}$$

where $0 < \delta \leq \dfrac{c_{n+1}}{(n+1)^p}$.

Proof. To prove (i) part, we define the function $\omega(z)$ by

$$\frac{1+\omega(z)}{1-\omega(z)} = \frac{c_{n+1}}{(n+1)^p \delta}\left[\frac{D^p f(z)}{D^p f_n(z)} - \left(\frac{c_{n+1} - (n+1)^p \delta}{c_{n+1}}\right)\right]$$

$$= \frac{1 + \sum\limits_{k=2}^{n} k^p a_k z^{k-1} + \frac{c_{n+1}}{(n+1)^p \delta} \sum\limits_{k=n+1}^{\infty} k^p a_k z^{k-1}}{1 + \sum\limits_{k=2}^{n} k^p a_k z^{k-1}}. \tag{10}$$

It suffices to show that $|\omega(z)| \leq 1$. Now, from (10) we can write

$$\omega(z) = \frac{\frac{c_{n+1}}{(n+1)^p \delta} \sum\limits_{k=n+1}^{\infty} k^p a_k z^{k-1}}{2 + 2\sum\limits_{k=2}^{n} k^p a_k z^{k-1} + \frac{c_{n+1}}{(n+1)^p \delta} \sum\limits_{k=n+1}^{\infty} k^p a_k z^{k-1}}.$$

Hence we obtain $|\omega(z)| \leq \dfrac{\frac{c_{n+1}}{(n+1)^p \delta} \sum\limits_{k=n+1}^{\infty} k^p |a_k|}{2 - 2\sum\limits_{k=2}^{n} k^p |a_k| - \frac{c_{n+1}}{(n+1)^p \delta} \sum\limits_{k=n+1}^{\infty} k^p |a_k|}.$

Now $|\omega(z)| \leq 1$ if $2\frac{c_{n+1}}{(n+1)^p\delta}\sum\limits_{k=n+1}^{\infty} k^p|a_k| \leq 2-2\sum\limits_{k=2}^{n} k^p|a_k|$ or, equivalently,

$$\sum_{k=2}^{n} k^p|a_k| + \frac{c_{n+1}}{(n+1)^p\delta}\sum_{k=n+1}^{\infty} k^p|a_k| \leq 1. \tag{11}$$

It suffices to show that the L.H.S. of (11) is bounded above by $\sum_{k=2}^{\infty}\frac{c_k}{\delta}|a_k|$, which is equivalent to

$$\sum_{k=2}^{n}\left(\frac{c_k-\delta k^p}{\delta}\right)|a_k| + \sum_{k=n+1}^{\infty}\left(\frac{(n+1)^p c_k - c_{n+1}k^p}{(n+1)^p\delta}\right)|a_k| \geq 0. \tag{12}$$

To see that the function given by (9) gives the sharp result we observe that for $z = re^{i\pi/n}$

$\frac{D^p f(z)}{D^p f_n(z)} = 1+\frac{\delta}{c_{n+1}}(n+1)^p z^n \to 1-\frac{\delta}{c_{n+1}}(n+1)^p = \dfrac{c_{n+1}-\delta(n+1)^p}{c_{n+1}}$, when $r \to 1^-$.

To prove the (ii) part of this theorem, we write

$$\frac{1+\omega(z)}{1-\omega(z)} = \frac{c_{n+1}+(n+1)^p\delta}{(n+1)^p\delta}\left[\frac{D^p f(z)}{D^p f_n(z)} - \left(\frac{c_{n+1}}{c_{n+1}+(n+1)^p\delta}\right)\right]$$

$$= \frac{1+\sum\limits_{k=2}^{n} k^p a_k z^{k-1} - \frac{c_{n+1}}{(n+1)^p\delta}\sum\limits_{k=n+1}^{\infty} k^p a_k z^{k-1}}{1+\sum\limits_{k=2}^{\infty} k^p a_k z^{k-1}}$$

where $|\omega(z)| \leq \dfrac{\left(\frac{c_{n+1}+(n+1)^p\delta}{(n+1)^p\delta}\right)\sum\limits_{k=n+1}^{\infty} k^p|a_k|}{2-2\sum\limits_{k=2}^{n} k^p|a_k| - \frac{c_{n+1}-(n+1)^p\delta}{(n+1)^p\delta}\sum\limits_{k=n+1}^{\infty} k^p|a_k|} \leq 1$. This last inequality is equivalent to $\sum\limits_{k=2}^{n} k^p|a_k| + \frac{c_{n+1}}{(n+1)^p\delta}\sum\limits_{k=n+1}^{\infty} k^p|a_k| \leq 1$. Making use of (6) to get (12). Finally, equality holds in (8) for the function $f(z)$ given by (9). This completes the proof of Theorem 2.1.

Taking $p=0$ in Theorem 2.1, we obtain the following result given by Frasin in [2].

Corollary 2.2. *If $f \in H_\phi(c_k,\delta)$, then*

$$Re\left\{\frac{f(z)}{f_n(z)}\right\} \geq \frac{c_{n+1}-\delta}{c_{n+1}}, \qquad (z \in U) \tag{13}$$

and

$$Re\left\{\frac{f_n(z)}{f(z)}\right\} \geq \frac{c_{n+1}}{c_{n+1}+\delta}, \qquad (z \in U) \tag{14}$$

where $c_k \geq \begin{cases} \delta & if \quad k=2,3,....,n \\ c_{n+1} & if \quad k=n+1,n+2... \end{cases}$. *The results* (13) *and* (14) *are sharp with the function given by* (9).

Taking $p=1$ in Theorem 2.1, we obtain

Corollary 2.3. *If* $f \in H_\phi(c_k,\delta)$ *, then*

$$Re\left\{\frac{f'(z)}{f'_n(z)}\right\} \geq \frac{c_{n+1}-(n+1)\delta}{c_{n+1}}, \qquad (z \in U) \tag{15}$$

and

$$Re\left\{\frac{f'_n(z)}{f'(z)}\right\} \geq \frac{c_{n+1}}{c_{n+1}+(n+1)\delta}, \qquad (z \in U), \tag{16}$$

where

$$c_k \geq \begin{cases} k\delta & if \quad k=2,3,....,n \\ \dfrac{kc_{n+1}}{n+1} & if \quad k=n+1,n+2... \end{cases} \tag{17}$$

The results (15) *and* (16) *are sharp with the function given by* (9).

Remark 2.1. Frasin has shown in Theorem 2.7 of [2] that for $f \in H_\phi(c_k,\delta)$, inequalities (15) and (16) hold with the condition

$$c_k \geq \begin{cases} k\delta & if \quad k=2,3,....,n \\ k\delta\left(1+\dfrac{c_{n+1}}{n+1}\right) & if \quad k=n+1,n+2... \end{cases} \tag{18}$$

But it can be easily seen that the condition (18) for $k=n+1$ gives $c_{n+1} \geq (n+1)\delta\left(1+\frac{c_{n+1}}{(n+1)\delta}\right)$ or, equivalently $\delta \leq 0$, which contradicts the initial assumption $\delta > 0$. So Theorem 2.7 of [2] does not seem suitable with the condition (18) and our condition (17) remedies this problem.

Taking $p=0$, $c_k = \dfrac{[(1+\beta)k-(\alpha+\beta)]}{1-\alpha}\begin{pmatrix} k+\lambda-1 \\ k \end{pmatrix}$, where $\lambda \geq 0$, $\beta \geq 0$, $-1 \leq \alpha < 1$ and $\delta = 1$ in Theorem 2.1, we obtain the following result given by Rosy et al. in [3].

Corollary 2.4. *If* f *of the form* (1) *and satisfy the condition* $\sum_{k=2}^{\infty} c_k |a_k| \leq 1$, *where* $c_k = \dfrac{[(1+\beta)k-(\alpha+\beta)]}{1-\alpha}\begin{pmatrix} k+\lambda-1 \\ k \end{pmatrix}$ *, where* $\lambda \geq 0$, $\beta \geq 0$, $-1 \leq \alpha < 1$.

Then

$$Re\left\{\frac{f(z)}{f_n(z)}\right\} \geq \frac{c_{n+1}-1}{c_{n+1}}, \qquad (z \in U) \tag{19}$$

and

$$Re\left\{\frac{f_n(z)}{f(z)}\right\} \geq \frac{c_{n+1}}{c_{n+1}+1}, \qquad (z \in U). \tag{20}$$

The results (19) *and* (20) *are sharp with the function given by*

$$f(z) = z + \frac{1}{c_{n+1}} z^{n+1}. \tag{21}$$

. Taking $p = 1, c_k = \frac{[(1+\beta)k-(\alpha+\beta)]}{1-\alpha}\begin{pmatrix} k+\lambda-1 \\ k \end{pmatrix}$, where $\lambda \geq 0,\ \beta \geq 0,\ -1 \leq \alpha < 1$ and $\delta = 1$ in Theorem 2.1, we obtain

Corollary 2.5. *If f of the form* (1) *and satisfy the condition* $\sum_{k=2}^{\infty} c_k |a_k| \leq 1$, *where* $c_k = \frac{[(1+\beta)k-(\alpha+\beta)]}{1-\alpha}\begin{pmatrix} k+\lambda-1 \\ k \end{pmatrix}$, $(\lambda \geq 0,\ \beta \geq 0,\ -1 \leq \alpha < 1)$.
Then

$$Re\left\{\frac{f'(z)}{f'_n(z)}\right\} \geq \frac{c_{n+1}-(n+1)}{c_{n+1}}, \qquad (z \in U) \tag{22}$$

and

$$Re\left\{\frac{f'_n(z)}{f'(z)}\right\} \geq \frac{c_{n+1}}{c_{n+1}+(n+1)}, \qquad (z \in U). \tag{23}$$

where

$$c_k \geq \begin{cases} k & if \quad k = 2,3,....,n \\ \frac{kc_{n+1}}{n+1} & if \quad k = n+1, n+2... \end{cases} \tag{24}$$

The results (22) *and* (23) *are sharp with the function given by* (21).
Taking $p = 0$, $c_k = \lambda_k - \alpha\mu_k$, $\delta = 1-\alpha$,where $0 \leq \alpha < 1$, $\lambda_k \geq 0$, $\mu_k \geq 0$, and $\lambda_k \geq \mu_k\,(k \geq 2)$ in Theorem 2.1, we obtain the following result given by Frasin in [1].
Corollary 2.6. *If f of the form* (1) *and satisfies the condition* $\sum_{k=2}^{\infty}(\lambda_k - \alpha\mu_k)|a_k| \leq 1-\alpha$, *then*

$$Re\left\{\frac{f(z)}{f_n(z)}\right\} \geq \frac{\lambda_{n+1}-\alpha\mu_{n+1}-1+\alpha}{\lambda_{n+1}-\alpha\mu_{n+1}}, \qquad (z \in U) \tag{25}$$

and

$$Re\left\{\frac{f_n(z)}{f(z)}\right\} \geq \frac{\lambda_{n+1}-\alpha\mu_{n+1}}{\lambda_{n+1}-\alpha\mu_{n+1}+1-\alpha}, \qquad (z \in U). \tag{26}$$

where

$$\lambda_k - \alpha\mu_k \geq \begin{cases} 1-\alpha & if \quad k = 2,3,....,n \\ \lambda_{n+1} - \alpha\mu_{n+1} & if \quad k = n+1, n+2... \end{cases}$$

The results (25) *and* (26) *are sharp with the function given by*

$$f(z) = z + \frac{1-\alpha}{\lambda_{n+1} - \alpha\mu_{n+1}} z^{n+1}. \tag{27}$$

Taking $p = 1$, $c_k = \lambda_k - \alpha\mu_k$, $\delta = 1 - \alpha$ where $0 \leq \alpha < 1$, $\lambda_k \geq 0$, $\mu_k \geq 0$, and $\lambda_k \geq \mu_k \, (k \geq 2)$ in Theorem 2.1, we obtain

Corollary 2.7. *If f of the form* (1) *and satisfies the condition* $\sum_{k=2}^{\infty} (\lambda_k - \alpha\mu_k) |a_k| \leq 1 - \alpha$ *then*

$$Re\left\{\frac{f'(z)}{f_n'(z)}\right\} \geq \frac{\lambda_{n+1} - \alpha\mu_{n+1} - (n+1)(1-\alpha)}{\lambda_{n+1} - \alpha\mu_{n+1}}, \qquad (z \in U) \tag{28}$$

and

$$Re\left\{\frac{f_n'(z)}{f'(z)}\right\} \geq \frac{\lambda_{n+1} - \alpha\mu_{n+1}}{\lambda_{n+1} - \alpha\mu_{n+1} + (n+1)(1-\alpha)}, \qquad (z \in U) \tag{29}$$

where

$$\lambda_k - \alpha\mu_k \geq \begin{cases} k(1-\alpha) & if \quad k = 2,3,....,n \\ \frac{k(\lambda_{n+1} - \alpha\mu_{n+1})}{n+1} & if \quad k = n+1, n+2... \end{cases} \tag{30}$$

The results (28) *and* (29) *are sharp with the function given by* (27).

Remark 2.2. Frasin has obtained inequalities (28) and (29) in Theorem 2 of [1] under condition

$$\lambda_{k+1} - \alpha\mu_{k+1} \geq \begin{cases} k(1-\alpha) & if \quad k = 2,3,....,n \\ k(1-\alpha) + \frac{k(\lambda_{n+1} - \alpha\mu_{n+1})}{n+1} & if \quad k = n+1, n+2... \end{cases} \tag{31}$$

.

But when we critically observe the proof of Theorem 2 of [1], we find that last inequality of this theorem

$$\sum_{k=2}^{n} \left(\frac{\lambda_k - \alpha\mu_k}{1-\alpha} - k\right)|a_k| + \sum_{k=n+1}^{\infty} \left(\frac{\lambda_k - \alpha\mu_k}{1-\alpha} - \left(1 + \frac{\lambda_{n+1} - \alpha\mu_{n+1}}{(n+1)(1-\alpha)}\right)k\right)|a_k| \geq 0. \tag{32}$$

We easily see that the inequality (32) of [[1] ,Theorem 2] can not be hold for the function given by (27) for supporting the sharpness of the results (28) and (29). So the condition 2.25 of Theorem 2 in [1] is incorrect and correct results are mentioned in Corollary 2.7.

Remark 2.3. Taking $p = 0$, $c_k = (k - \alpha)$, $c_k = k(k - \alpha)$, $\delta = 1 - \alpha$, $0 \leq \alpha < 1$ in Theorem 2.1, we obtain Theorem 1-3 given by Silverman in [6].

Remark 2.4. Taking $p = 1$, $c_k = (k - \alpha)$, $c_k = k(k - \alpha)$, $\delta = 1 - \alpha$, $0 \leq \alpha < 1$ in Theorem 2.1, we obtain Theorem 4-5 given by Silverman in [6].

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