Acta Universitatis Apulensis No. 26/2011
ISSN: 1582-5329 pp. 29-34

OPEN MAPPING THEOREM IN 2-NORMED SPACE

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Abstract. In this paper, we give some properties of the sets $B_e(a,r)$ and $B_e[a,r]$. This enables us to obtain a result analogue of "Open mapping theorem"for 2-normed space

2000 *Mathematics Subject Classification*: 41A65, 41A15.

Keywords: Linear 2-normed space, compact operator, absolutely convex, locally bounded, open set.

1. Introduction

The concept of 2-normed spaces was introduced and studied by Siegfried Gähler, a German Mathematician who worked at German Academy of Science, Berlin, in a series of paper in German language published in Mathematische Nachrichten. This notion which is nothing but a two dimensional analogue of a normed space got the attention of a wider audience after the publication of a paper by Albert George, White Jr. of USA in 1969 entitled 2-Banach spaces [11]. In the same year Gähler published another paper on this theme in the same journal [5]. A.H. Siddiqi delivered a series of lectures on this theme in various conferences in India and Iran. His joint paper with S. Gähler and S.C.Gupta [4] of 1975 also provide valuable results related to the theme of this paper.Results up to 1977 were summarized in the survey paper by A.H. Siddiqi [10]. In the recent years, a number of articles devoted to 2-normed space have been published. Several authors try to obtain new applications of these notions in some abstract settings.The first and second part of the paper consists of introduction and preliminaries. In Section 3, one learns about the main purpose of the paper. The authors are going to construct Open mapping Theorem for a real linear 2-normed spaces. The scheme of proofs are similar to that known in the case of the classical thoery for normed spaces. In [7] S. Gähler introduced the following definition of a 2-normed space.

2.Preliminaries

Definition 2.1[7]. *Let X be a real linear space of dimension greater than 1. Suppose $\|\ ,\ \|$ is a real valued function on $X \times X$ satisfying the following conditions:*

1. $\|x, y\| = 0$ *if and only if x and y are linearly independent*
2. $\|x, y\| = \|y, x\|$
3. $\|\alpha x, y\| = |\alpha|\|x, y\|$
4. $\|x + y, z\| \leq \|x, z\| + \|y, z\|$

Then $\|\ ,\ \|$ is called a 2-norm on X and the pair $(X, \|\ ,\ \|)$ is called a 2-normed space. Some of the basic properties of 2-norms, that they are non-negative and $\|x, y + x\| = \|x, y\|, \forall x, y \in X$ and $\forall \alpha \in$ R

Definition 2.2[7]. *A sequence $\{x_n\}$ in a 2-normed space X is said to be convergent if there exists an element $x \in X$ such that*

$$\lim_{n\to\infty} \|\{x_n - x\}, y\| = 0$$

for all $y \in X$.

Definition 2.3 [7]. *Let X and Y be two 2-normed spaces and $T : X \to Y$ be a linear operator. The operator T is said to be sequentially continuous at $x \in X$ if for any sequence $\{x_n\}$ of X converging to x, we have $T(\{x_n\}) \to T(x)$.*

Definition 2.4 [2]. *The closure of a subset E of a 2-normed space X is denoted by $\overline{E}$ and defined by the set of all $x \in X$ such that there is a sequence $\{x_n\}$ of E converging to x. We say that E is closed if $E = \overline{E}$.*

For a 2 normed space we consider the subsets

$$B_e(a, r) = \{x : \|x - a, e\| < r\}.$$

$$B_e[a, r] = \{x : \|x - a, e\| \leq r\}.$$

Definition 2.5 [2]. *Let X be a linear space. A subset C of X is called convex(resp. absolutely convex) if $\alpha C + \beta C \subseteq C$ for every $\alpha, \beta > 0$(resp. $\alpha, \beta \in K$) with $\alpha + \beta = 1$(resp.$|\alpha| + |\beta| \leq 1$)*

Remark 1. A subset C of a vector space X is absolutely convex iff it is convex and balanced.

Proposition 2.6[2].*Let X and Y be linear spaces and $T : X \to Y$ a linear map. Let A be a convex(resp. absolutely convex) subset of X and B be a convex(resp.*

absolutely convex) subset of Y. Then $T(A)$ is convex(resp. absolutely convex) subset of Y and $T^{-1}(B)$ is convex(resp. absolutely convex) subset of X

3.Main Results

Proposition 3.1. *Let X be a 2-normed space. Then the subsets*

$$B_e(0,r) = \{x : \|x,e\| < r\}$$

$$B_e[0,r] = \{x : \|x,e\| \leq r\}$$

are absolutely convex for every $r > 0$ and $e \in X$

Proof. Let $x,\ y \in B_e(0,r)$ and let α , $\beta \in K$ with $|\alpha| + |\beta| \leq 1$.
Then,

$$\begin{aligned} \|\alpha x + \beta y, e\| &\leq \|\alpha x, e\| + \|\beta y, e\| \\ &= |\alpha|\|x,e\| + |\beta|\|y,e\| \\ &< (|\alpha| + |\beta|)r \\ &\leq r. \end{aligned}$$

This implies that $\alpha x + \beta y \in B_e(0,r)$. Therefore $B_e(0,r)$ is absolutely convex. Similarly $B_e[0,r]$ is absolutely convex.

Proposition 3.2. *The closure of a convex (resp.absolutely convex) subset of a 2-normed space is convex (resp.absolutely convex)*

Proof. Let X be a 2-normed space and C be a convex (resp.absolutely convex)subset of X. Let $x,\ y \in \overline{C}$ and $\alpha, \beta > 0$(resp. $\alpha, \beta \in K$) with $\alpha + \beta = 1$(resp.$|\alpha| + |\beta| \leq 1$). Then there exist sequences $\{x_n\}$++ and $\{y_n\}$ in C such that $x_n \to x$ and $y_n \to y$. Since C is convex(resp.absolutely convex), $\alpha x_n + \beta y_n \in C$for every $n \in \mathbb{N}$, so that $\alpha x + \beta y = \lim_{n\to\infty}(\alpha x_n + \beta y_n) \in \overline{C}$. Hence $\overline{C}$ is convex (resp.absolutely convex).

Definition 3.3. *Let X be a 2-normed space. A subset F of X is said to be open in X if for every $a \in F$ there exist some $e \in X$ and $r > 0$ such that $B_e(a,r) \subseteq F$.*

Remark 2. $B_e(a,r)$ is open.

For every $a' \in B_e(a,r)$, let $r' = r - \|a' - a, e\|$. Then $B_e(a',r') \subseteq B_e(a,r)$

Remark 3. E is closed iff its complement E' is open.

Suppose that E is closed. Let $a \in E'$. Then for some $e \in X$ and $r > 0$, $B_e(a,r) \cap E =$

$\emptyset$.If not let $B_e(a,r) \cap E \neq \emptyset$ for every $e \in X$ and $r > 0$, then for $r = \frac{1}{n}$ there exist a sequence $\{x_n\}$ in E such that $\|x_n - a, e\| < \frac{1}{n}$, $\forall\ n \in \mathbb{N}$ and $e \in X$.This implies that $x_n \to a$ and therefore $a \in \overline{E} = E$, a contradiction. Conversely if E' is open, consider the sequence $\{x_n\}$ in E such that $x_n \to a$. If $a \notin E$ then there exist $B_e(a,r)$ such that $B_e(a,r) \subseteq E'$ for some $e \in X$ and $r > 0$. As $x_n \to a$, $\|x_n - a, x\| \to 0$, $\forall\ x \in X$. Therefore there exist some K such that $\|x_K - a, x\| < r$. This implies $x_K \in B_e(a,r) \subseteq E'$, a contradiction.

Definition 3.4. *Let X be a 2-normed space. A point $a \in E$ is called an interior point of E if there exist some $e \in X$ and $r > 0$ such that $B_e(a,r) \subseteq E$.*

Definition 3.5. *A 2-normed space X is called a Baire space if each non empty open subset of X is second category in X.*

Proposition 3.6.*Let X be a 2-normed Baire space and C be an absolutely convex closed subset of X such that $X = \bigcup_{n=1}^{\infty} nC$. Then 0 is an interior point of C*

Proof. Since X is a Baire space, there exist some $n \in \mathbb{N}$ such that nC has non-empty interior. Hence there exist some $e \in X$ and $r > 0$ such that $B_e(x,r) \subseteq C$. This implies that $B_e(-x,r) \subseteq -C \subseteq C$. For any $y \in B_e(0,r)$,

$$y = \frac{1}{2}(x+y) + \frac{1}{2}(-x+y) \subseteq \frac{1}{2}C + \frac{1}{2}C \subseteq C.$$

Therefore, $B_e(0,r) \subseteq C$. Hence 0 is an interior point of C.

Proposition 3.7.*Let X be a 2-normed space, Y a 2-normed Baire space and $T : X \to Y$ a surjective linear map. Then 0 is an interior point of $\overline{T\left(B_e[0,1]\right)}$*

Proof. $\overline{T\left(B_e[0,1]\right)}$ is absolutely convex (Propostions 2.6, 3.1 and 3.2) and

$$\begin{aligned} \bigcup_{n=1}^{\infty} n\overline{T\left(B_e[0,1]\right)} \supseteq \bigcup_{n=1}^{\infty} nT\left(B_e[0,1]\right) &= \bigcup_{n=1}^{\infty} T\left(nB_e[0,1]\right) \\ = T\left(\bigcup_{n=1}^{\infty} nB_e[0,1]\right) &= T(X) \\ &= Y \end{aligned}$$

$\Rightarrow$ $$Y = \bigcup_{n=1}^{\infty} n\overline{T\left(B_e[0,1]\right)}$$

Therefore by proposition 3.6, 0 is an interior point of $\overline{T\left(B_e[0,1]\right)}$.

Proposition 3.8. *Let X be a 2-Banach space, Y a 2-normed space and let $T : X \to Y$ be a sequentially continuous map. If 0 is an interior point of $\overline{T\left(B_e[0,1]\right)}$ then 0 is an interior point of $T\left(B_e[0,1]\right)$*

Proof. By hypothesis there exist some $e' \in Y$ and $r > 0$ such that

$$B_{e'}[0,r] \subseteq \overline{T\left(B_e[0,1]\right)}$$

Let $y \in B_{e'}[0, \frac{r}{2}]$. Choose $x_1, x_2, ..x_n \in B_e[0,1]$ such that

$$\begin{aligned}
\|y - T\left(\sum_{m=1}^{n-1} \frac{x_m}{2^m}\right), z\| &< \frac{r}{2^n}, \forall\, n \in \mathbb{N} \quad and\; z \in Y \\
\Rightarrow \qquad \|y - T\left(\sum_{m=1}^{n-1} \frac{x_m}{2^m}\right), e'\| &< \frac{r}{2^n}, \forall\, n \in \mathbb{N} \\
i.e. y - T\left(\sum_{m=1}^{n-1} \frac{x_m}{2^m}\right) \in B_{e'}[0, \frac{r}{2^n}] &= \frac{1}{2^n} B_{e'}[0, r] \subseteq \frac{1}{2^n}\overline{T\left(B_e[0,1]\right)}
\end{aligned}$$

$\Rightarrow$ There exist a sequence $\{x_n\}$ in $B_e[0,1]$ such that

$$\begin{aligned}
\|y - T\left(\sum_{m=1}^{n-1} \frac{x_m}{2^m}\right) - \frac{1}{2^n} T(x_n), z\| &< \frac{r}{2^{n+1}},\ \forall\, z \in Y \; and\; n \in \mathbb{N}. \\
i.e. \quad \|y - T\left(\sum_{m=1}^{n} \frac{x_m}{2^m}\right), z\| &< \frac{r}{2^{n+1}},\ \forall\, z \in Y \; and\; n \in \mathbb{N}.
\end{aligned}$$

Also,

$$\|\sum_{m=1}^{n} \frac{x_m}{2^m}, e\| \leq \sum_{m=1}^{n} \frac{\|x_m, e\|}{2^m} \leq \sum_{m=1}^{n} \frac{1}{2^m} \ \forall\, n \in \mathbb{N}.$$

$$\Rightarrow \qquad \|\sum_{m=1}^{\infty} \frac{x_m}{2^m}, e\| \leq 1$$

Let $x = \sum_{m=1}^{\infty} \frac{x_m}{2^m}$. Then,

$$y = \lim_{n \to \infty} T\left(\sum_{m=1}^{n} \frac{x_m}{2^m}\right) = T(x) \in T\left(B_e[0,1]\right)$$

Hence $B_{e'}[0, \frac{r}{2}] \subseteq T\left(B_e[0,1]\right)$.

Theorem 3.9(Open Mapping Theorem). *Every surjective sequentially continuous map from a 2-Banach space onto a 2-normed Baire space is open, i.e. maps open sets into open sets*

Proof. Let $T : X \to Y$ be a surjective sequentially continuous map from a 2-Banach space X onto a 2-normed Baire space Y. Let U be an open subset of X and let $y \in T(U)$. Take $a \in U$ with $T(a) = y$. Then there exist some $e \in X$ and $r > 0$ such that $B_e[a, r] \subseteq U$.

$\Rightarrow \quad a + rB_e[0,1] \subseteq U$ and therefore $T(a + rB_e[0,1]) \subseteq T(U)$
$\Rightarrow \quad y + rT(B_e[0,1]) \subseteq T(U)$
Since 0 is an interior point of $T(B_e[0,1])$, there exist some $e' \in Y$ and $r' > 0$ such that $B_{e'}(y, r') \subseteq T(U)$. Hence $T(U)$ is open.

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