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DOMINATION NUMBERS OF COMPLETE $P_{12} \times P_N$ GRID GRAPH

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Abstract. this paper concerns the domination numbers $\gamma(P_{12} \times P_n)$ for $n \geq 1$. these numbers were proviously established for $1 \leq n \leq 33$ [2] . the domination set decision problem is NP-complete [3] , [5]

Key words: dominating set, domination numbers, cartesian product of two paths.

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1. Introduction

Two vertices u and v of a graph $G = P_{12} \times P_n$, are said to be adjacent if $uv \in E$. The neighborhood of $v \in G$, is the set of vertices of G which are adjacent to v, the neighborhood of v is denoted by $N(v)$. The closed neighborhood of v is $\overline{N}(v)$, $\overline{N}(v) = N(v) \sqcup \{v\}$ [1] .

The degree of a vertex v is the cardinality of $N(v)$; *i.e.*
$d(v) = |N(v)|$.

A dominating set in a graph is a set of vertices having the property that every vertex not in the set is adjacent to a vertex in the set.

The domination numbers $\gamma(P_{12} \times P_n)$ is the cardinality of a smallest dominating set in $P_{12} \times P_n$.

2. Definitions

Let D be a dominating set of a graph $P_{12} \times P_n = (V, E)$.

1. We define the function C_D , which we call the weight function, as follows :

$C_D : V \to \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers, $C_D(v) = \left|\widetilde{N}(v)\right|$, where $\widetilde{N}(v) = \{w \in D : vw \in E \text{ or } w = v\}$

i.e. the weight of v is the number of vertices in D which dominates v.

2. We say that v, $v \in D$, has a moving domination if and only if one of the following two cases occurs :

(a) For every vertex w, $w \in N(v)$, $C_D(w) \geq 2$. And, hence, the domination of v can be transformed to any vertex in $N(v) - D$.

(b) there exists one vertex u, $u \in N(v)$, such that $C_D(u) = 1$.

In this case, the domination of v can be transformed only to u.

3. We say that a vertex v, $v \in D$, is a redundant vertex of D if $C_D(w) \geq 2$ for every vertex $w \in \overline{N}(v)$.

4. If $v \in D$, which has a moving domination, we say that v is inefficient if transforming the domination from v to any vertex in $N(v)$ would not produce any redundant vertex.

3.Complete $P_{12} \times P_n$ grid graph:

for two vertices v_0 and v_n of a graph $P_{12} \times P_n$, a $v_0 - v_n$ walk is a alternating sequence of vertices and edges v_0, e_1, v_1, ...,e_n, v_n such that consecutive vertices and edges are incident.

A path is a walk in which no vertex is repeated.

A path with n vertices is denoted by P_n , it has $n-1$ edges, the lenght is $n-1$; the cartesian product $P_k \times P_n$ of two path is the complete grid graph with vertex set $V = \{(i,j) : 1 \leq i \leq k \ , \ 1 \leq j \leq n\}$, where $(u_1, u_2)(v_1, v_2)$ is a edge of $P_k \times P_n$ if $|u_1 - v_1| + |u_2 - v_2| = 1$ [4] .

4.An algorithm for finding a dominating set of a graph $P_{12} \times P_n$ using a transformation of domination of vertices

1. Let $P_{12} \times P_n = (V, E)$, $|V| = m$, $m > 1$.

2. Let $D = V$ be a dominating set of $P_{12} \times P_n$.

3. Pick a vertex v_1 of D, and delete from D all vertices w, $w \in N(v_1)$.

Then, for $1 < n < \frac{m}{2}$, pick a vertex v_n , $v_n \in D - \overset{n-1}{\underset{i=1}{\sqcup}} \overline{N}(v_i)$ and delete from D all vertices w, $w \in N(v_n) - \overset{n-1}{\underset{i=1}{\sqcup}} \overline{N}(v_i)$.

4. If D contains a redundant vertex, then delete it. Repeated this proces until D has no redundant vertex.

5. Transform domination from vertices of D which have moving domination to vertices in $V - D$ to obtain redundant vertices and go to step 4.

If no redundant vertex can be obtained by a transformation of domination of vertices of D, then stop, and the obtained dominating set D satisfies :

For every $v \in D$, $\exists w \in \overline{N}(v) : C_D(w) = 1$.

Example No.1

1. Let (k,n) be the vertex in the $k-th$ row and in the $n-th$ column of the graph $P_{12} \times P_{13}$, $|V| = 156$.

2. Let $D = V$, dominating set of $P_{12} \times P_{13}$.

3. Pick a vertex $v_1 = (1,1) \in D$, and delete from D all vertices w, $w \in N(v_1)$, then, for $1 < n < \frac{156}{2}$, pick a vertex v_n , $v_n \in D - \overline{\bigsqcup_{i=1}^{n-1} N}(v_i)$, and delete from D all vertices w, $w \in N(v_n) - \overline{\bigsqcup_{i=1}^{n-1} N}(v_i)$. We obtain the dominating set D (black circles) in figure 1.

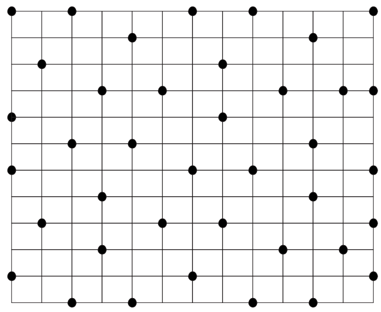


Figure 1.

4. Since for every vertex $v \in D$, $\exists w \in \overline{N}(v)$ such that $C_D(w) = 1$, D has no redundant vertices.

5. Transform the domination from the vertex $(9,13)$ to the vertex $(9,12)$ and delete, from D, the resullting redundant vertex $(10,12)$.

Therefore, the set D indicated in figure 2 (black circles) is a dominating set of $P_{12} \times P_{13}$.

Note that D minimum dominating set (see [2]). $\gamma(P_{12} \times P_{13}) = 38$

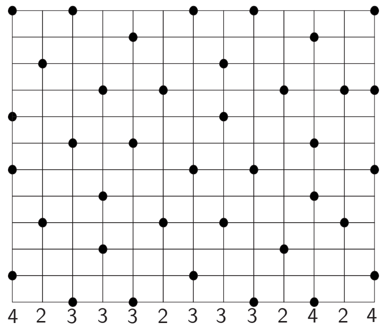
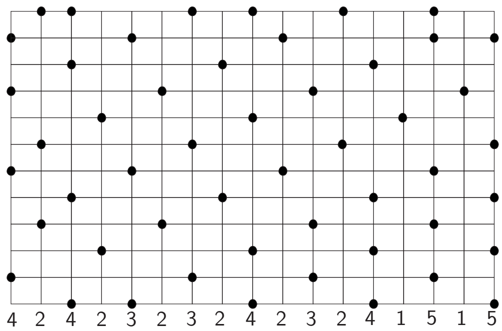


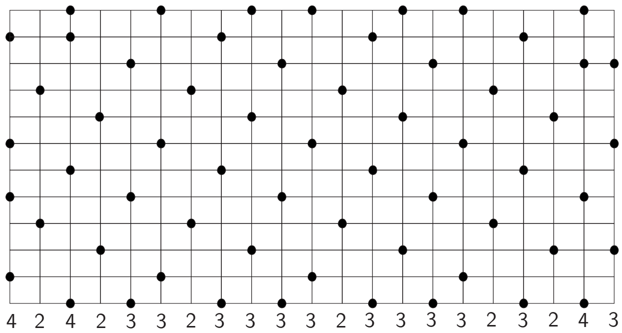
Figure 2.

$n = 13$, $\gamma(P_{12} \times P_{13}) = 38 = 3(n-7) + 20 = 3n - 1$
and gradually have : $\gamma(P_{12} \times P_n) = 3n - 1 \quad for \quad 9 \leq n \leq 13$

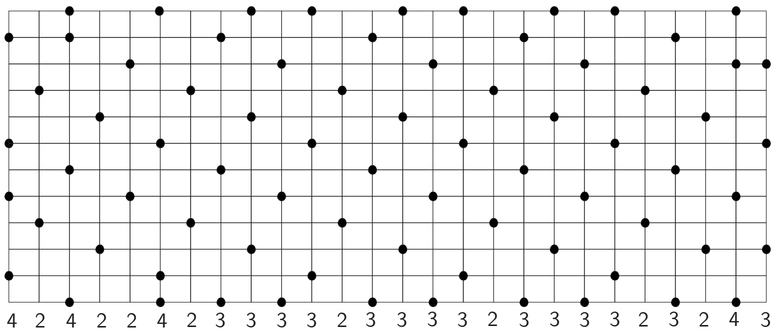
So, we gradually get domination numbers of $P_{12} \times P_n$.



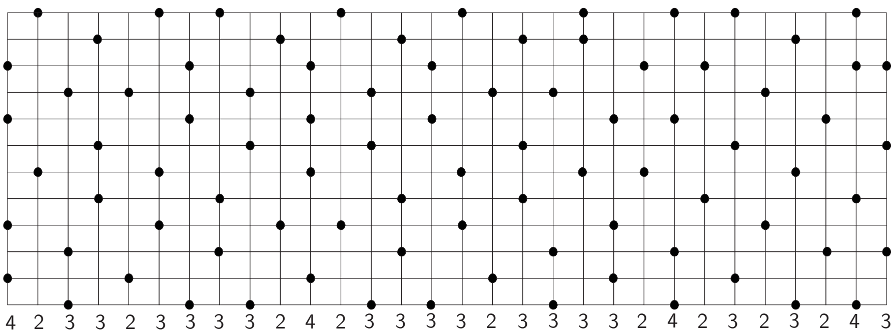
$n = 17$, $\gamma(P_{12} \times P_{17}) = 49 = 3(n-14) + 40 = 3n - 2$
and gradually have : $\gamma(P_{12} \times P_n) = 3n - 2 \quad for \quad 14 \leq n \leq 17$



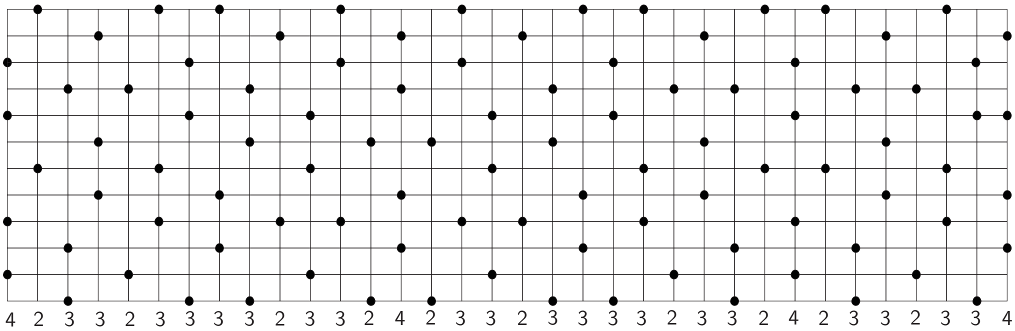
$n = 21$, $\gamma(P_{12} \times P_{21}) = 60 = 3(n-9) + 24 = 3n - 3$
and gradually have : $\gamma(P_{12} \times P_n) = 3n - 3 \;\; for \;\; 18 \leq n \leq 21$



$n = 26$, $\gamma(P_{12} \times P_{26}) = 74 = 3(n-12) + 32 = 3n - 4$
and gradually have : $\gamma(P_{12} \times P_n) = 3n - 4 \;\; for \;\; 22 \leq n \leq 26$



$n = 30$, $\gamma(P_{12} \times P_{30}) = 85 = 3(n-13) + 34 = 3n - 5$
and gradually have : $\gamma(P_{12} \times P_n) = 3n - 5 \;\; for \;\; 27 \leq n \leq 30$



$n = 34$, $\gamma(P_{12} \times P_{34}) = 96 = 3(n-14) + 36 = 3n - 6$
and gradually have : $\gamma(P_{12} \times P_n) = 3n - 6 \quad for \quad 31 \leq n \leq 34$

So, we gradually have :

$$\gamma(P_{12} \times P_n) = \begin{cases} 3n & for \quad n = 4, 6, 7, 8 \\ 3n+1 & for \quad n = 1, 2, 3, 5 \\ 3n - (3t+1) & for \quad n = 9 + 13t, 10 + 13t, ..., 13 + 13t; t \geq 0 \\ 3n - (2 + t + 3k) & for \quad n = 14 + 4t + 13k, ..., 17 + 4t + 13k, \\ & \qquad \begin{cases} t = 0, 1 \\ k \geq 0 \end{cases} \end{cases}$$

where k and t are a integers.

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