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APPLICATIONS OF SUBORDINATION ON SUBCLASSES OF MEROMORPHICALLY UNIVALENT FUNCTIONS WITH INTEGRAL OPERATOR

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Abstract. In this paper we are concerned with applications of differential subordination for class of meromorphic univalent functions defined by integral operator,

$$P_\beta^\alpha f(z) = \frac{\beta^\alpha}{\Gamma(\alpha)} . \frac{1}{z^{\beta+1}} \int_0^z t^\beta \Big(\log \frac{z}{t}\Big)^{\alpha-1} f(t)dt$$

In the present paper, our aim is to study the Coefficient Bounds, Integral Representation, Linear Combinations, Weighted and Arithmetic Mean.

Keywords: Meromorphic Functions, Differential Subordination, Integral Operator, Coefficient Bounds, Integral Representation, Linear Combination, Weighted Means and Arithmetic Means.

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1. Introduction

Let Σ be a class of all Meromorphic functions $f(z)$ of the form by [4]

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k, \qquad \mathrm{a}_k \geq 0 \tag{1.1}$$

and

$$g(z) = \frac{1}{z} + \sum_{k=1}^{\infty} b_k z^k, \qquad \mathrm{b}_k \geq 0 \tag{1.2}$$

Which are univalent in the punctured unit disk $U = \{z : z \in \mathbb{C} \ , 0 < |z| < 1\} = U \setminus \{0\}$ with a simple pole at the origin. If $f(z)$ and $g(z)$ are analytic in U, we say that $f(z)$ is subordinate to $g(z)$, written $f \prec g$ if there exists a Schwarz function $w(z)$ in U with $w(0) = 0$ and $|w(z)| < 1 (z \in U)$, such that $f(z) = g(w(z)), (z \in U)$. In particular, if the function $g(z)$ is univalent in U, we have the following[5],

$$f(z) \prec g(z), \ \ (z \in U) \Leftrightarrow f(0) = g(0) \ and \ f(U) \subset g(U)$$

Definition 1. A*nalogous to the operators defined by Jung, Kim, and Srivastava[3] on the normalized Analytic functions, by [1] define the following integral operator*

$$P_\beta^\alpha : \sum \to \sum$$

$$P_\beta^\alpha = P_\beta^\alpha f(z) = \frac{\beta^\alpha}{\Gamma(\alpha)} . \frac{1}{z^{\beta+1}} \int_0^z t^\beta \Big(\log \frac{z}{t}\Big)^{\alpha-1} f(t)dt \tag{1.3}$$

$$(\alpha > 0, \beta > 0;\ z \in U)$$

Where $\Gamma(\alpha)$ is the familiar Gamma Function.
Using the integral representation of the Gamma and Beta function, it can be shown that
For $f(z) \in \Sigma$, given by (1.1) we have

$$P_\beta^\alpha f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \left(\frac{\beta}{k+\beta+1}\right)^\alpha a_k z^k, \qquad (\alpha > 0, \beta > 0) \tag{1.4}$$

$$z(P_\beta^\alpha f(z))' = \beta P_\beta^{\alpha-1} f(z) - (\beta+1) P_\beta^\alpha f(z), \qquad (\alpha > 1, \beta > 0) \tag{1.5}$$

Definition 2. *Let A and B $(-1 \le B < A \le 1)$ be defined parameters. We say that a function $f(z) \in \Sigma$ is in the class $\Sigma(A,B)$; if it satisfies the following subordination condition by [5]*

$$-z^2\big(p_\beta^\alpha f(z)\big)' \prec \frac{1+Az}{1+Bz} \qquad (z \in U) \tag{1.6}$$

By the definition of differential subordinate, (1.6) is equivalent to the following condition

$$\left| \frac{1 + z^2\Big(p_\beta^\alpha f(z)\Big)'}{A + Bz^2\Big(p_\beta^\alpha f(z)\Big)'} \right| < 1 \qquad (z \in U) \tag{1.7}$$

In particular, we can write

$$\sum(1-2\beta, -1) = \sum(\beta)$$

Where $\Sigma(\beta)$ denotes the class of function in Σ satisfying following form

$$Re\left(-z^2\left(p^{\alpha}_{\beta}f(z)\right)'\right) > \beta \qquad (0 \leq \beta < 1; z \in U)$$

Here are some applications of differential subordination {[2],[6]}.

2. Coefficient Bounds

Theorem 2.1 *Let the function $f(z)$ of the form (1.1) be in Σ Then the function $f(z)$ belongs to the class $\Sigma(A, B)$ if and only if*

$$(1-B)\sum_{k=1}^{\infty} k\left(\frac{\beta}{k+\beta+1}\right)^{\alpha} a_k < (A-B) \tag{2.1}$$

Where $-1 \leq B < A \leq 1$. The result is sharp for the function $f(z)$ given by

$$f(z) = \frac{1}{z} + \frac{(A-B)}{(1-B)k\left(\frac{\beta}{k+\beta+1}\right)^{\alpha}} z^k$$

Proof: Assume that the condition (2.1) is true. We must show that $f \in \Sigma(A, B)$ or equivalently prove that

$$\left|\frac{1+z^2\left(p^{\alpha}_{\beta}f(z)\right)'}{A+Bz^2\left(p^{\alpha}_{\beta}f(z)\right)'}\right| < 1 \tag{2.2}$$

$$\left|\frac{1+z^2\left(p^{\alpha}_{\beta}f(z)\right)'}{A+Bz^2\left(p^{\alpha}_{\beta}f(z)\right)'}\right| = \left|\frac{1+\left(-1+\sum\limits_{k=1}^{\infty} k\left(\frac{\beta}{k+\beta+1}\right)^{\alpha} a_k z^{k+1}\right)}{A+B\left(-1+\sum\limits_{k=1}^{\infty} k\left(\frac{\beta}{k+\beta+1}\right)^{\alpha} a_k z^{k+1}\right)}\right|$$

$$= \left|\frac{\sum\limits_{k=1}^{\infty} k\left(\frac{\beta}{k+\beta+1}\right)^{\alpha} a_k z^{k+1}}{A-B+B\sum\limits_{k=1}^{\infty} k\left(\frac{\beta}{k+\beta+1}\right)^{\alpha} a_k z^{k+1}}\right|$$

$$\leq \left|\frac{\sum\limits_{k=1}^{\infty} k\left(\frac{\beta}{k+\beta+1}\right)^{\alpha} a_k}{A-B+B\sum\limits_{k=1}^{\infty} k\left(\frac{\beta}{k+\beta+1}\right)^{\alpha} a_k}\right| < 1,$$

The last inequality is true by (2.1).

Conversely, suppose that $f \in \Sigma(A, B)$. We must show that the condition (2.1) holds true. We have

$$\left| \frac{1 + z^2 \left(p_\beta^\alpha f(z)\right)'}{A + Bz^2 \left(p_\beta^\alpha f(z)\right)'} \right| < 1$$

Hence we get

$$\left| \frac{\sum\limits_{k=1}^{\infty} k \left(\frac{\beta}{k+\beta+1}\right)^\alpha a_k}{A - B + B \sum\limits_{k=1}^{\infty} k \left(\frac{\beta}{k+\beta+1}\right)^\alpha a_k} \right| < 1,$$

Since $Re(z) < |z|$, so we have

$$\text{Re} \left\{ \frac{\sum\limits_{k=1}^{\infty} k \left(\frac{\beta}{k+\beta+1}\right)^\alpha a_k}{A - B + B \sum\limits_{k=1}^{\infty} k \left(\frac{\beta}{k+\beta+1}\right)^\alpha a_k} \right\} < 1$$

We Choose the value of z on the real axis and letting $z \to 1^-$, then we obtain

$$\left\{ \frac{\sum\limits_{k=1}^{\infty} k \left(\frac{\beta}{k+\beta+1}\right)^\alpha a_k}{A - B + B \sum\limits_{k=1}^{\infty} k \left(\frac{\beta}{k+\beta+1}\right)^\alpha a_k} \right\} < 1$$

Then

$$(1 - B) \sum_{k=1}^{\infty} k \left(\frac{\beta}{k + \beta + 1}\right)^\alpha a_k < (A - B)$$

The result is sharp for the function

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{(A - B)(k + \beta + 1)^\alpha}{(1 - B)\, k\beta^\alpha} z^k$$

Corollary 2.2 *Let $f \in \Sigma(A, B)$, then we have*

$$a_k \leq \frac{(A - B)(k + \beta + 1)^\alpha}{(1 - B)\, k(\beta)^\alpha} \qquad k \geq 1$$

3. Integral Representation

In the next theorem we obtain an integral representation for $p_\beta^\alpha f(z)$

Theorem 3.1 *Let $f \in \Sigma(A,B)$, then*

$$p_\beta^\alpha f(z) = \int_0^z \frac{(A\phi(t)-1)}{t^2\,(1-B\phi(t))}dt \quad , \qquad where\,|\phi(z)|<1, \quad z \in U \tag{3.1}$$

Proof: Let $f(z) \in \Sigma(A,B)$ letting
$-z^2\Big(p_\beta^\alpha f(z)\Big)' = y(z)$ We have

$$y(z) \prec \frac{1+Az}{1+Bz} \tag{3.2}$$

Or we can write

$$\left|\frac{y(z)-1}{By(z)-A}\right| < 1,$$

so that consequently, we have

$$\frac{y(z)-1}{By(z)-A} = \phi(z), \qquad |\phi(z)|<1 \qquad (z \in U)$$

We can write

$$-z^2\big(p_\beta^\alpha f(z)\big)' = \frac{1-A\phi(z)}{1-B\phi(z)},$$

Which gives

$$-\big(p_\beta^\alpha f(z)\big)' = \frac{1}{z^2}\frac{1-A\phi(z)}{1-B\phi(z)},$$

$$p_\beta^\alpha f(z) = \int_0^z \frac{1}{t^2}\frac{A\phi(t)-1}{1-B\phi(t)}dt \tag{3.3}$$

And this gives the required result.

4. Linear Combination

In the theorem below, we prove a linear combination for the class $\Sigma(A,B)$.

Theorem 4.1 *Let $f_i(z) = z^{-1} + \sum\limits_{k=1}^{\infty} a_{k,i}z^k, \qquad (a_{k,i} \geq 0, i = 1,2,....,l)$*

Belong to $\Sigma(A,B)$ *then*
$F(z)=\sum\limits_{i=1}^{l} c_i f_i(z) \in \Sigma(A,B)$ *Where* $\sum\limits_{i=1}^{l} c_i = 1$

Proof: By theorem 2.1, We can write for every $i \in \{1,2\ldots,l\}$

$$\sum_{k=1}^{\infty} \frac{k\,(1-B)}{(A-B)}\left(\frac{\beta}{k+\beta+1}\right)^{\alpha} a_{k,i} < 1, \tag{4.1}$$

Therefore

$$F(z)=\left(\sum_{i=1}^{l} c_i\left(z^{-1}+\sum_{k=1}^{\infty} a_{k,i}z^{k}\right)\right) \qquad = z^{-1}+\sum_{i=1}^{l}\sum_{k=1}^{\infty} c_i a_{k,i} z^{k}$$

$$= z^{-1}+\sum_{k=1}^{\infty}\left(\sum_{i=1}^{l} c_i a_{k,i}\right) z^{k} \tag{4.2}$$

however

$$\sum_{k=1}^{\infty} \frac{k\,(1-B)}{(A-B)}\left(\frac{\beta}{k+\beta+1}\right)^{\alpha}\left(\sum_{i=1}^{l} a_{k,i}c_i\right)$$

$$=\sum_{i=1}^{l}\left[\sum_{k=1}^{\infty} \frac{k\,(1-B)}{(A-B)}\left(\frac{\beta}{k+\beta+1}\right)^{\alpha} a_{k,i}\right] c_i$$

$$\leq 1 \tag{4.3}$$

then $F(z) \in \Sigma(A,B)$
hence the proof is complete.

5. Weighted Mean

Definition 3. *$f(z)$ and $g(z)$ belong to Σ ,then the weighted mean $h_j(z)$ of $f(z)$ and $g(z)$ is given by*

$$h_j(z)=\frac{1}{2}\left[(1-j)f(z)+(1+j)g(z)\right]$$

In the following theorem we will show the weighted mean for the class $\Sigma(A,B)$.

Theorem 5.1 *If $f(z)$ and $g(z)$ are in the class $\Sigma(A,B)$, then the weighted mean of*

$f(z)$ and $g(z)$ is also in $\Sigma(A,B)$

Proof: We have for $h_j(z)$ by definition

$$h(z) = \frac{1}{2}\left[(1-j)\left(z^{-1}+\sum_{k=1}^{\infty} a_k z^k\right) + (1+j)\left(z^{-1}+\sum_{k=1}^{\infty} b_k z^k\right)\right]$$

$$= z^{-1} + \frac{1}{2}\sum_{k=1}^{\infty}[(1-j)a_k + (1+j)b_k]z^k \tag{5.1}$$

Since $f(z)$ and $g(z)$ are in the class $\Sigma(A,B)$ so by theorem 2.1 we must prove that

$$\sum_{k=1}^{\infty} k(1-B)\left(\frac{\beta}{k+\beta+1}\right)^{\alpha}\left[\frac{1}{2}(1-j)a_k + \frac{1}{2}(1+j)b_k\right]$$

$$= \frac{1}{2}(1-j)(1-B)\sum_{k=1}^{\infty} k\left(\frac{\beta}{k+\beta+1}\right)^{\alpha} a_k + \frac{1}{2}(1+j)(1-B)\sum_{k=1}^{\infty} k\left(\frac{\beta}{k+\beta+1}\right)^{\alpha} b_k$$

$$\leq \frac{1}{2}(1-j)(A-B) + \frac{1}{2}(1+J)(A-B)$$

$$\leq (A-B) \tag{5.2}$$

hence proved

6. Arithmetic Mean

Definition 4. *Let $f_1(z)$,$f_2(z)\ldots f_l(z)$ belong to $\Sigma(A,B)$, then the arithmetic mean $h(z)$ of $f_i(z)$ is given by*

$$h(z) = \frac{1}{l}\sum_{k=1}^{l} f_i(z)$$

In the theorem below we will prove the arithmetic mean for this class $\Sigma(A,B)$.

Theorem 6.1 *If $f_1(z)$,$f_2(z)\ldots f_l(z)$ are in the class $\Sigma(A,B)$, then the arithmetic mean $h(z)$ of $f_i(z)$ is given by*

$$h(z) = \frac{1}{l}\sum_{i=1}^{l} f_i(z) \tag{6.1}$$

is also in the class $\Sigma(A, B)$.
Proof: We have for $h(z)$ by def. 4

$$h(z) = \frac{1}{l} \sum_{k=1}^{l} \left(z^{-1} + \sum_{k=1}^{\infty} a_{k,i} z^k \right) = z^{-1} + \sum_{k=1}^{\infty} \left(\frac{1}{l} \sum_{k=1}^{l} a_{k,i} \right) z^k \tag{6.2}$$

Since $f_i(z) \in \Sigma(A, B)$ for every $i = 1, 2, \ldots, l$ so by using theorem 2.1, we prove that

$$\begin{aligned} &(1-B) \sum_{k=1}^{\infty} k \left(\frac{\beta}{k+\beta+1} \right)^{\alpha} \left(\frac{1}{l} \sum_{i=1}^{l} a_{k,i} \right) \\ &= \frac{1}{l} \sum_{i=1}^{l} \left(\sum_{k=1}^{\infty} k(1-B) \right) \left(\frac{\beta}{k+\beta+1} \right)^{\alpha} a_{k,i} \\ &\leq \frac{1}{l} \sum_{i=1}^{l} (A-B) \end{aligned} \tag{6.3}$$

The proof is complete.

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