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A SUBCLASS OF SĂLĂGEAN - TYPE HARMONIC UNIVALENT FUNCTIONS

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Abstract. In this paper, we define and investigate a subclass of Salagean-type harmonic univalent functions. We obtain coefficient conditions, extreme points, distortion bounds, convex combination and radius of convexity for the above class of harmonic univalent functions.

Keywords: Harmonic univalent functions, Salagean operator.

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1.Introduction

A continuous complex-valued function $f = u+iv$ is defined in a simply-connected complex domain D is said to be harmonic in D if both u and v are real harmonic in D. In any simply-connected domain we can write

$$f = h + \overline{g} \; , \tag{1.1}$$

where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h^{'}(z)| > |g^{'}(z)|$ in D (see [2]).

Denote by S_H the class of functions f of the form (1.1) that are harmonic univalent and sense-preserving in the unit disc $U = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \overline{g} \in S_H$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k \, z^k \, , g(z) = z + \sum_{k=1}^{\infty} b_k \, z^k \; , \; |b_1| < 1 \; . \tag{1.2}$$

In 1984 Clunie and Shell-Small [2] investigated the class S_H as well as its geometric subclasses and obtained some coefficient bounds. Since then, there have been several related papers on S_H and its subclasses.

For $f = g + h$ given by (1.2), Jahangiri et al. [4] defined the modified Salagean operator of f as

$$D^m f(z) = D^m h(z) + (-1)^m \overline{D^m g(z)}, \tag{1.3}$$

where

$$D^m h(z) = z + \sum_{k=2}^{\infty} k^m \, a_k \, z^k \quad \text{and} \quad D^m g(z) = \sum_{k=1}^{\infty} k^m \, b_k \, z^k \ .$$

The differential operator D^m was introduced by Salagean [5].

For $0 \leq \alpha < 1, 0 \leq \lambda \leq 1, m \in N = \{1, 2, ...\}, n \in N_0 = N \cup \{0\}, m > n$ and $z \in U$, we let $S_H(m, n; \alpha; \lambda)$ denote the family of harmonic functions f of the form (1.2) such that

$$Re \left\{ \frac{D^m \ f(z)}{\lambda \, D^m f(z) + (1-\lambda) D^n \, f(z)} \right\} > \alpha \ , \tag{1.4}$$

where $D^m f$ is defined by (1.3).

We let the subclass $\overline{S}_H(m, n; \alpha; \lambda)$ consist of harmonic functions $f_m = h + \overline{g}_m$ in $\overline{S}_H(m, n; \alpha; \lambda)$ so that h and g_m are of the form

$$h(z) = z - \sum_{k=2}^{\infty} a_k \, z^k \, , g_m(z) = (-1)^{m-1} \sum_{k=1}^{\infty} b_k \, z^k \ ; a_k \, , \, b_k \geq 0 \ . \tag{1.5}$$

We note that, by the special choices of m, n, α and λ, we obtain the following classes studied by various authors:

(i) $\overline{S}_H(1, 0, 0; 0) = T_H^{*0}$, the class of sense-preserving, harmonic univalent functions f which are starlike in U, $\overline{S}_H(2, 1; 0; 0) = K_H^0$, the class of sense-preserving, harmonic univalent functions f which are convex in U, studied by Silverman [6];

(ii) $\overline{S}_H(1, 0; \alpha, 0) = \Im_H(\alpha)$, the class of sense-preserving, harmonic univalent functions f which are starlike of order α in U, $\overline{S}_H(2, 1; \alpha; 0) = K_H(\alpha)$, the class of sense-preserving, harmonic univalent functions f which are convex of order α in U, studied by Jahangiri [3];

(iii) $\overline{S}_H(n + 1, n; \alpha; 0) = \overline{H}(n, \alpha)$, the class of Salagean-type harmonic univalent functions studied by Jahangiri et al. [4].

(iv) $\overline{S}_H(m, n; \alpha, 0) = \overline{S}_H(m, n; \alpha)$, is a new class of Salagean-type harmonic univalent functions, studied by Yaclin [8].

We further, observe that, by the special choices of m, n, α and λ our class $\overline{S}_H(m, n; \alpha; \lambda)$ gives rise to the following new subclasses of S_H:

(i) $\overline{S}_H(1, 0; \alpha; \lambda) = \Im_H(\alpha, \lambda)$

$$= \left\{ f \in S_H : Re \left\{ \frac{\frac{zf'(z)}{f(z)}}{\lambda \frac{z\,f'(z)}{f(z)} + (1-\lambda)} \right\} > \alpha \, , 0 \leq \alpha < 1, \, 0 \leq \lambda \leq 1, \ \ z \in U \right\},$$

(ii) $S_H(2,0;\alpha;\lambda) = K_H(\alpha,\lambda)$

$$= \left\{ f \in S_H : Re \left\{ \frac{1 + \frac{zf''(z)}{f'(z)}}{\lambda(1 + \frac{zf''(z)}{f'(z)}) + (1-\lambda)} \right\} > \alpha\,, 0 \le \alpha < 1,\ 0 \le \lambda \le 1,\ \ z \in U \right\},$$

(iii) $S_H(n+1,n;\alpha;\lambda) = S_H(n;\alpha;\lambda)$

$$= \left\{ f \in S_H : Re \left\{ \frac{\frac{D^{n+1}f(z)}{D^n f(z)}}{\lambda \frac{D^{n+1}(z)}{D^n(z)} + (1-\lambda)} \right\} > \alpha\,, 0 \le \alpha < 1,\ 0 \le \lambda \le 1,\ \ n \in N_0\,,\ z \in U \right\}.$$

We let the subclasses $\overline{\Im}_H(\alpha,\lambda), \overline{K}_H(\alpha,\lambda)$ and $\overline{S}_H(n;\alpha;\lambda)$ consist of harmonic functions $f_m = h + \overline{g}_m$ so that h and g_m are of the form (1.5).

For the harmonic functions f of the form (1.2) with $b_1 = 0$, Avic and Zlotkiewicz [1] showed that if $\sum\limits_{k=2}^{\infty} k^2(|a_k| + |b_k|) \le 1$ then $f \in K_H^0$, and Silverman [6] proved that the above coefficient condition is also necessary if $f = h + \overline{g}$ has negative coefficients. Later Silverman and Silvia [7] improved the results of [1, 6] to the case b_1 not necessarily zero.

For the harmonic functions f_m of the form (1.5) Yalcin [8] showed that $f_m \in \overline{S}_H(m,n;\alpha)$ if and only if $\sum\limits_{k=1}^{\infty} (\frac{k^m - \alpha k^n}{1-\alpha}\, a_k + \frac{k^m - (-1)^{m-n}\,\alpha\, k^n}{1-\alpha} b_k) \le 2$. In this paper we extend the above results to the classes $S_H(m,n;\alpha;\lambda)$ and $\overline{S}_H(m,n;\alpha;\lambda)$. We also obtain coefficient conditions, extreme points, distortion bounds, convolution conditions and convex combinations for $\overline{S}_H(m,n;\alpha;\lambda)$.

2. Coefficient characterization

Unless otherwise mentioned, we assume throughout this paper that $m \in N,\ n \in N_0,\ m > n,\ 0 \le \alpha < 1$ and $0 \le \lambda \le \frac{1-\alpha}{1+\alpha}$. We begin with a sufficient condition for functions in $S_H(m,n;\alpha;\lambda)$.

Theorem 1. *Let $f = h + \overline{g}$ be so that h and g given by* (1.2)*. Furthermore, let*

$$\sum_{k=1}^{\infty} \left\{ \frac{(1-\lambda\alpha)k^m - \alpha(1-\lambda)k^n}{1-\alpha}\ |a_k| + \frac{(1-\lambda\alpha)k^m - (-1)^{m-n}\alpha(1-\lambda)k^n}{1-\alpha}\ |b_k| \right\} \le 2\,, \tag{2.1}$$

where $a_1 = 1$, $m \in N$, $n \in N_0$, $m > n$, $0 \le \alpha < 1$ and $0 \le \lambda \le \frac{1-\alpha}{1+\alpha}$. Then f is sense-preserving, harmonic univalent in U and $f \in S_H(m,n;\alpha;\lambda)$.

Proof. If $z_1 \neq z_2$, then

$$\begin{aligned}\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| &\geq 1-\left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right| \\ &= 1-\left|\frac{\sum\limits_{k=1}^{\infty} b_k(z_1^k-z_2^k)}{(z_1-z_2)+\sum\limits_{k=2}^{\infty} a_k\,(z_1^k-z_2^k)}\right|\end{aligned}$$

$$\begin{aligned}&> 1-\frac{\sum\limits_{k=1}^{\infty} k\,|b_k|}{1-\sum\limits_{k=2}^{\infty} k\;|a_k|} \\ &\geq 1-\frac{\sum\limits_{k=1}^{\infty} \frac{(1-\lambda\alpha)k^m-(-1)^{m-n}\alpha(1-\lambda)k^n}{1-\alpha}\;|b_k|}{1-\sum\limits_{k=2}^{\infty} \frac{(1-\lambda\alpha)k^m\;-\alpha(1-\lambda)k^n}{1-\alpha}\;|a_k|}\;\geq 0,\end{aligned}$$

which proves univalence. Note that f is sense-preserving in U. This is because

$$\begin{aligned}|h^{'}(z)| &\geq 1-\sum_{k=2}^{\infty} k\;|a_k|\;|z|^{k-1} > 1-\sum_{k=2}^{\infty} k\,|a_k| \\ &\geq 1-\sum_{k=2}^{\infty} \frac{(1-\lambda\alpha)k^m-\alpha(1-\lambda)k^n}{1-\alpha}\;|a_k| \\ &\geq \sum_{k=1}^{\infty} \frac{(1-\lambda\alpha)k^m-(-1)^{m-n}\alpha(1-\lambda)k^n}{1-\alpha}\;|b_k| \\ &> \sum_{k=1}^{\infty} \frac{(1-\lambda\alpha)k^m-(-1)^{m-n}\alpha(1-\lambda)k^n}{1-\alpha}\;|b_k|\;|z|^{k-1} \\ &\geq \sum_{k=1}^{\infty} k\,|b_k|\;|z|^{k-1} \geq |g^{'}(z)|\;.\end{aligned}$$

Now we show that $f \in S_H(m,n;\alpha;\lambda)$. We only need to show that if (2.1) holds then the condition (1.4) is satisfied.

Using the fact that $Re\,w > \alpha$ if and only if $|1-\alpha+w| > |1+\alpha-w|$, it suffices to show that

$$|D^m f(z) + (1-\alpha)[\lambda\, D^m f(z) + (1-\lambda)\, D^n f(z)]| -$$

$$|D^m f(z) - (1+\alpha)[\lambda\, D^m f(z) + (1-\lambda)\, D^n f(z)]| > 0\ . \qquad (2.2)$$

Substituting for $D^m f(z)$ and $D^n f(z)$ in (2.2) yields, by (2.1) and $0 \le \lambda \le \frac{1-\alpha}{1+\alpha}$, we obtain

$$\begin{aligned}
& |(1+\lambda(1-\alpha))D^m f(z) + (1-\lambda)(1-\alpha)D^n f(z)| - \\
& |(1-\lambda(1+\alpha))D^m f(z) - (1-\lambda)(1+\alpha)D^n f(z)| \\
= & \left|(2-\alpha)z + \sum_{k=2}^{\infty} [(1-\alpha)(1-\lambda)k^n + (1+\lambda(1-\alpha))k^m]\, a_k\, z^k \right. \\
& \left. +(-1)^n \sum_{k=1}^{\infty} \left[(1-\alpha)(1-\lambda)k^n + (-1)^{m-n}(1+\lambda(1-\alpha))k^m\right] \overline{b_k\, z^k}\right| \\
& -\left|\alpha\ z - \sum_{k=2}^{\infty} [(1-\lambda(1+\alpha))k^m - (1+\alpha)(1-\lambda)k^n]\, a_k\, z^k \right. \\
& \left. -(-1)^n \sum_{k=1}^{\infty} \left[(-1)^{m-n}(1-\lambda(1+\alpha))k^m - (1+\alpha)(1-\lambda)k^n\right] \overline{b_k\, z^k}\right| \\
\ge & \ 2(1-\alpha)|z| - 2\sum_{k=2}^{\infty} [(1-\lambda\alpha)k^m - \alpha(1-\lambda)k^n]\, |a_k|\ |z|^k \\
& -\sum_{k=1}^{\infty} \left|(-1)^{m-n}(1+(1-\alpha)\lambda)k^m + (1-\alpha)(1-\lambda)k^n\right|\ |b_k|\ |z|^k \\
& -\sum_{k=1}^{\infty} \left|(-1)^{m-n}(1-\lambda(1+\alpha))k^m - (1+\alpha)(1-\lambda)k^n\right|\ |b_k|\ |z|^k
\end{aligned}$$

$$= \begin{cases}
2(1-\alpha)|z| - 2\sum\limits_{k=2}^{\infty} [(1-\lambda\alpha)k^m - \alpha(1-\lambda)k^n]\, |a_k|\ |z|^k - & \\
2\sum\limits_{k=1}^{\infty} [(1-\lambda\alpha)k^m + \alpha\,(1-\lambda)k^n]\, |b_k|\, |z|^k, & m-n \quad \text{is odd} \\
2(1-\alpha)|z| - 2\sum\limits_{k=2}^{\infty} [(1-\lambda\alpha)k^m - \alpha(1-\lambda)k^n]\, |a_k|\ |z|^k & \\
-2\sum\limits_{k=1}^{\infty} [(1-\lambda\alpha)k^m - \alpha\,(1-\lambda)k^n]\, |b_k|\, |z|^k, & m-n \quad \text{is even}
\end{cases}$$

$$= 2(1-\alpha)|z| \left\{ \begin{array}{l} 1 - \sum\limits_{k=2}^{\infty} \frac{(1-\lambda)k^m - \alpha(1-\lambda)k^n}{1-\alpha}\ |a_k|\ |z|^{k-1} \\ - \sum\limits_{k=1}^{\infty} \frac{(1-\lambda\alpha)k^m - (-1)^{m-n}\alpha(1-\lambda)k^n}{1-\alpha}\ |b_k|\ |z|^{k-1} \end{array} \right\}$$

$$> 2(1-\alpha)\left\{\begin{array}{l} 1-\left(\sum\limits_{k=2}^{\infty}\frac{(1-\lambda\alpha)k^m-(-1)^{m-n}\alpha(1-\lambda)k^n}{1-\alpha}\,|a_k|\right. \\ \left.+\sum\limits_{k=1}^{\infty}\frac{(1-\lambda\alpha)k^m-(-1)^{m-n}\alpha(1-\lambda)k^n}{1-\alpha}\,|b_k|\right) \end{array}\right\}.$$

This last expression is non-negative by (2.1).

The harmonic univalent functions

$$f(z) = z + \sum_{k=2}^{\infty} \frac{1-\alpha}{(1-\lambda\alpha)k^m - \alpha(1-\lambda)k^n}\, x_k\, z^k +$$

$$\sum_{k=1}^{\infty} \frac{1-\alpha}{(1-\lambda\alpha)k^m - (-1)^{m-n}\alpha(1-\lambda)k^n}\, \overline{y_k\, z^k}\,, \tag{2.3}$$

where $m \in N$, $n \in N_0$, $m > n$, $0 \le \alpha < 1$, $0 \le \lambda \le \frac{1-\alpha}{1+\alpha}$ and $\sum\limits_{k=2}^{\infty} |x_k| + \sum\limits_{k=1}^{\infty} |y_k| = 1$, show that the coefficient bound given by (2.1) is sharp. The functions of the form (2.3) are in $S_H(m, n; \alpha; \lambda)$ because

$$\sum_{k=1}^{\infty}\left(\frac{(1-\lambda\alpha)k^m - \alpha(1-\lambda)k^n}{1-\alpha}\,|a_k| + \frac{(1-\lambda\alpha)k^m - (-1)^{m-n}\alpha(1-\lambda)k^n}{1-\alpha}\,|b_k|\right)$$

$$= 1 + \sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| = 2.$$

This completes the proof of Theorem 1.

In the following theorem, it is shown that the condition (2.1) is also necessary for functions $f_m = h + \overline{g}_m$, where h and g_m are of the form (1.5).

Theorem 2. *Let* $f_m = h + \overline{g}_m$ *be given by* (1.5). *Then* $f_m \in \overline{S}_H(m, n; \alpha; \lambda)$ *if and only if*

$$\sum_{k=1}^{\infty} \left\{[(1-\lambda\alpha)k^m - \alpha(1-\lambda)k^n]\, a_k + [(1-\lambda\alpha)k^m - \right.$$

$$\left.(-1)^{m-n}\alpha(1-\lambda)k^n] b_k\right\} \le 2(1-\alpha), \tag{2.4}$$

where $a_1 = 1$, $m \in N$, $n \in N_0$, $m > n$ and $0 \le \alpha < 1$, $0 \le \lambda \le \frac{1-\alpha}{1+\alpha}$.

Proof. Since $\overline{S}_H(m, n; \alpha; \lambda) \subset S_H(m, n; \alpha; \lambda)$, we only need to prove the "only if" part of the theorem. To this end, for functions f_m of the form (1.5), we notice that the condition $Re\left\{\frac{D^m f_m(z)}{\lambda D^m f_m(z)+(1-\lambda)D^n f_m(z)}\right\} > \alpha$ is equivalent to

$$Re\left\{\frac{(1-\alpha)z-\sum_{k=2}^{\infty}[(1-\alpha\lambda)k^m-\alpha(1-\lambda)k^n]a_k\, z^k+ (-1)^{2m-1}\sum_{k=1}^{\infty}[(1-\alpha\lambda)k^m-(-1)^{m-n}\alpha(1-\lambda)k^n]b_k\,\overline{z}^k}{z-\sum_{k=2}^{\infty}(\lambda k^m+(1-\lambda)k^n)a_k\, z^k+\sum_{k=1}^{\infty}((-1)^{m-1}\lambda\, k^m+(-1)^{m+n-1}(1-\lambda)k^n)b_k\,\overline{z}^k}\right\}\geq 0\,. \quad (2.5)$$

The above required condition (2.5) must hold for all values of z in U. Upon choosing the values of z on the positive real axis where $0\leq z=r<1$, we must have

$$\frac{(1-\alpha)-\sum_{k=2}^{\infty}[(1-\alpha\lambda)k^m-\alpha(1-\lambda)k^n]a_k\, r^{k-1}-\sum_{k=1}^{\infty}[(1-\alpha\lambda)k^m-(-1)^{m-n}\alpha(1-\lambda)k^n]b_k\, r^{k-1}}{1-\sum_{k=2}^{\infty}(\lambda k^m+(1-\lambda)k^n)a_k\, r^{k-1}-\sum_{k=1}^{\infty}(\lambda k^m+(-1)^{m-n}(1-\lambda)k^n)b_k\, r^{k-1}}\geq 0\,. \quad (2.6)$$

If the condition (2.4) does not hold, then the numerator in (2.6) is negative for r sufficiently close to 1. Hence there exists $z_0=r_0$ in $(0,1)$ for which the quotient in (2.6) is negative. This contradicts the required condition for $f_m\in\overline{S}_H(m,n;\alpha;\lambda)$ and so the proof is complete.

3.Extreme points and distortion theorem

Our next theorem is on the extreme points of convex hulls of $\overline{S}_H(m,n;\alpha;\lambda)$ denoted by clco $\overline{S}_H(m,n;\alpha;\lambda)$.

Theorem 3. *Let f_m be given by* (1.5). *Then $f_m\in\overline{S}_H(m,n;\alpha;\lambda)$ if and only if $f_m(z)=\sum_{k=1}^{\infty}(x_k\, h_k(z)+y_k g_{m_k}(z))$, where $h_1(z)=z$,*

$$h_k(z)=z-\frac{1-\alpha}{(1-\lambda\alpha)k^m-\alpha(1-\lambda)k^n}\; z^k\ (k=2,3,...),$$

and

$$g_{m_k}(z)=z+(-1)^{m-1}\frac{1-\alpha}{(1-\lambda\alpha)k^m-(-1)^{m-n}\alpha(1-\lambda)k^n}\;\overline{z}^k\,,$$

$$(k=1,2,...), x_k\geq 0,\ y_k\geq 0\,, x_1=1-\sum_{k=2}^{\infty}(x_k+y_k)\geq 0\ .$$

In particular, the extreme points of $\overline{S}_H(m,n;\alpha;\lambda)$ are $\{h_k\}$ and $\{g_{m_k}\}$.

Proof. Suppose

$$\begin{aligned} f_m(z) &= \sum_{k=1}^{\infty}(x_k\, h_k(z) + y_k g_{m_k}(z)) \\ &= \sum_{k=1}^{\infty}(x_k + y_k)z - \sum_{k=2}^{\infty}\frac{1-\alpha}{(1-\lambda\alpha)k^m - \alpha(1-\lambda)k^n}\ x_k\ z^k \\ &\quad +(-1)^{m-1}\sum_{k=1}^{\infty}\frac{1-\alpha}{(1-\lambda\alpha)k^m - (-1)^{m-n}\alpha(1-\lambda)k^n}\ y_k\ \overline{z}^k\ . \end{aligned}$$

Then

$$\begin{aligned} &\sum_{k=2}^{\infty}\frac{(1-\lambda\alpha)k^m - \alpha(1-\lambda)k^n}{1-\alpha}\ .\ \left(\frac{1-\alpha}{(1-\lambda\alpha)k^m - \alpha(1-\lambda)k^n}\ x_k\ \right) + \\ &\sum_{k=1}^{\infty}\frac{(1-\lambda\alpha)k^m - (-1)^{m-n}\alpha(1-\lambda)k^n}{1-\alpha}\ .\ \left(\frac{1-\alpha}{(1-\lambda\alpha)k^m - (-1)^{m-n}\alpha(1-\lambda)k^n}\ y_k\ \right) \\ = &\sum_{k=2}^{\infty}x_k\ +\ \sum_{k=1}^{\infty}y_k = 1 - x_1 \le 1 \end{aligned}$$

and so $f_m \in \overline{S}_H(m,n;\alpha;\lambda)$.

Conversely, if $f_m \in clco\ \overline{S}_H(m,n;\alpha;\lambda)$; then

$$a_k \le \frac{1-\alpha}{(1-\lambda\alpha)k^m - \alpha(1-\lambda)k^n}$$

and

$$b_k \le \frac{1-\alpha}{(1-\lambda\alpha)k^m - (-1)^{m-n}\alpha(1-\lambda)k^n}\ .$$

Set

$$x_k = \frac{(1-\lambda\alpha)k^m - \alpha(1-\lambda)k^n}{1-\alpha}\ ,\ (k = 2,3,....),$$

and

$$y_k = \frac{(1-\lambda\alpha)k^m - (-1)^{m-n}\alpha(1-\lambda)k^n}{1-\alpha}\ b_k,\ (k = 1,2,....).$$

Then note that by Theorem 2, $0 \le x_k \le 1$, $(k = 2,3,...)$, and $0 \le y_k \le 1$, $(k = 1,2,...)$. We define $x_1 = 1 - \sum\limits_{k=2}^{\infty} x_k - \sum\limits_{k=1}^{\infty} y_k$ and note that by Theorem 2, $x_1 \ge 0$. Consequently, we obtain $f_m(z) = \sum\limits_{k=1}^{\infty}(x_k\, h_k(z) + y_k\ g_k(z))$ as required.

The following theorem gives the distortion bounds for functions in $\overline{S}_H(m,n;\alpha;\lambda)$ which yields a covering result for this class.

Theorem 4. *Let* $f_m(z) \in \overline{S}_H(m,n;\alpha;\lambda)$. *Then for* $|z| = r < 1$, *we have*

$$|f_m(z)| \le (1+b_1)\,r + \frac{1}{2^n}\,\Big(\frac{1-\alpha}{(1-\lambda\alpha)2^{m-n}-\alpha(1-\lambda)}$$

$$-\frac{(1-\lambda\alpha)-(-1)^{m-n}\alpha(1-\lambda)}{(1-\lambda\alpha)2^{m-n}-\alpha(1-\lambda)}\,b_1\Big)\,r^2 \ (|z| = r < 1),$$

and

$$|f_m(z)| \ge (1-b_1)\,r - \frac{1}{2^n}\,\Big(\frac{1-\alpha}{(1-\lambda\alpha)2^{m-n}-\alpha(1-\lambda)}$$

$$-\frac{(1-\lambda\alpha)-(-1)^{m-n}\alpha(1-\lambda)}{(1-\lambda\alpha)2^{m-n}-\alpha(1-\lambda)}\,b_1\Big)\,r^2 \ (|z| = r < 1).$$

Proof. We only prove the right-hand inequality. The proof for the left-hand inequality is similar and will be omitted. Let $f_m(z) \in \overline{S}_H(m,n;\alpha;\lambda)$. Taking the absolute value of f_m we have

$$\begin{aligned}
|f_m(z)| &\le (1+b_1)\,r + \sum_{k=2}^{\infty}(a_k+b_k)\,r^k \le \quad (1+\mathrm{b}_1)r + \sum_{k=2}^{\infty}(a_k+b_k)\,r^2 \\
&= (1+b_1)r + \frac{1-\alpha}{2^n[(1-\lambda\alpha)2^{m-n}-\alpha(1-\lambda)]}\,\cdot \\
&\quad \cdot\sum_{k=2}^{\infty}\frac{2^n[(1-\lambda)2^{m-n}-\alpha(1-\lambda)]}{1-\alpha}\,(a_k+b_k)\,r^2 \\
&\le (1+b_1)r + \frac{(1-\alpha)r^2}{2^n[(1-\lambda\alpha)2^{m-n}-\alpha(1-\lambda)]}\,\cdot \\
&\quad \cdot\sum_{k=2}^{\infty}\Bigg[\frac{(1-\lambda\alpha)k^m-\alpha(1-\lambda)k^n}{1-\alpha}\,a_k + \\
&\quad \frac{(1-\lambda\alpha)k^m-(-1)^{m-n}\alpha(1-\lambda)k^n}{1-\alpha}\,b_k\Bigg] \\
&\le (1+b_1)r + \frac{1}{2^n}\Bigg[\frac{1-\alpha}{(1-\lambda\alpha)2^{m-n}-\alpha(1-\lambda)} - \\
&\quad \frac{(1-\lambda\alpha)-(-1)^{m-n}\alpha(1-\lambda)}{(1-\lambda\alpha)2^{m-n}-\alpha(1-\lambda)}\,b_1\Bigg]\,r^2\,.
\end{aligned}$$

The bounds given in Theorem 4 for functions $f_m = h+\overline{g}_m$ of form (1.5) also hold for functions of the form (1.2) if the coefficient condition (2.1) is satisfied. The upper bound given for $f \in \overline{S}_H(m,n;\alpha;\lambda)$ is sharp and the equality occurs for the functions

$$f(z) = z + b_1\,\overline{z} - \frac{1}{2^n}\left(\frac{1-\alpha}{(1-\lambda\alpha)2^{m-n}-\alpha(1-\lambda)} - \frac{(1-\lambda\alpha)-(-1)^{m-n}\alpha(1-\lambda)}{(1-\lambda\alpha)2^{m-n}-\alpha(1-\lambda)}\,b_1\right)\overline{z}^2,$$

and

$$f(z) = z - b_1\,\overline{z} - \frac{1}{2^n}\left(\frac{1-\alpha}{(1-\lambda\alpha)2^{m-n}-\alpha(1-\lambda)} - \frac{(1-\lambda\alpha)-(-1)^{m-n}\alpha(1-\lambda)}{(1-\lambda\alpha)2^{m-n}-\alpha(1-\lambda)}\,b_1\right)z^2$$

for $b_1 \le \frac{1-\alpha}{(1-\lambda\alpha)-(-1)^{m-n}\alpha(1-\lambda)}$ show that the bounds given in Theorem 4 are sharp.

The following covering result follows from the left hand inequality in Theorem 4.

Corollary 1. *Let the function* f_m *defined by* (1.5) *belong to the class* $\overline{S}_H(m,n;\alpha;\lambda)$. *Then*

$$\left\{w : |w| < \frac{(1-\lambda\alpha)2^m - 1 - [(1-\lambda)2^n - 1]\alpha}{(1-\lambda\alpha)2^m - \alpha(1-\lambda)2^n} - \frac{(1-\lambda\alpha)(2^m-1) - \alpha(1-\lambda)(2^n-(-1)^{m-n})}{(1-\lambda\alpha)2^m - \alpha(1-\lambda)2^n}\,b_1\right\} \subset f_m(U).$$

3.Convolution and convex combination

For our next theorem, we need to define the convolution of two harmonic functions.

For harmonic functions of the form:

$$f_m(z) = z - \sum_{k=2}^{\infty} a_k\,z^k + (-1)^{m-1}\sum_{k=1}^{\infty} b_k\ \overline{z}^k \quad (a_k \ge 0;\ b_k \ge 0) \tag{4.1}$$

and

$$F_m(z) = z - \sum_{k=2}^{\infty} A_k\,z^k + (-1)^{m-1}\sum_{k=1}^{\infty} B_k\ \overline{z}^k \quad (A_k \ge 0;\ B_k \ge 0) \tag{4.2}$$

we define the convolution of two harmonic functions f_m and F_m as

$$(f_m * F_m)(z) = f_m(z) * F_m(z)$$

$$= z - \sum_{k=2}^{\infty} a_k \, A_k \, z^k + (-1)^{m-1} \sum_{k=1}^{\infty} b_k \, B_k \, \overline{z}^k \, . \tag{4.3}$$

Using this definition, we show that the class $\overline{S}_H(m,n;\alpha;\lambda)$ is closed under convolution.

Theorem 5. *For* $0 \le \beta \le \alpha < 1, 0 \le \lambda \le \frac{1-\alpha}{1+\alpha}$*, let* $f_m \in \overline{S}_H(m,n;\alpha;\lambda)$ *and* $F_m \in \overline{S}_H(m,n;\beta;\lambda)$*. Then* $\overline{S}_H(m,n;\alpha;\lambda) \subset \overline{S}_H(m,n;\beta,\lambda)$.

Proof. Let the function $f_m(z)$ defined by (4.1) be in the class $\overline{S}_H(m,n;\alpha;\lambda)$ and let the function $F_m(z)$ defined by (4.2) be in the class $\overline{S}_H(m,n;\beta;\lambda)$. Then the convolution $f_m * F_m$ is given by (4.3). We wish to show that the coefficients of $f_m * F_m$ satisfy the required condition given in Theorem 2. For $F_m \in \overline{S}_H(m,n;\beta;\lambda)$ we note that $0 \le A_k \le 1$ and $0 \le B_k \le 1$. Now, for the convolution function $f_m * F_m$ we obtain

$$\begin{aligned}
& \sum_{k=2}^{\infty} \frac{(1-\lambda\beta)k^m - \beta(1-\lambda)k^n}{1-\beta} a_k \, A_k + \sum_{k=1}^{\infty} \frac{(1-\lambda\beta)k^m - (-1)^{m-n}\beta(1-\lambda)k^n}{1-\beta} b_k \, B_k \\
\le \; & \sum_{k=2}^{\infty} \frac{(1-\lambda\beta)k^m - \beta(1-\lambda)k^n}{1-\beta} a_k + \sum_{k=1}^{\infty} \frac{(1-\lambda\beta)k^m - (-1)^{m-n}\beta(1-\lambda)k^n}{1-\beta} b_k \\
\le \; & \sum_{k=2}^{\infty} \frac{(1-\lambda\alpha)k^m - \alpha(1-\lambda)k^n}{1-\alpha} a_k + \sum_{k=1}^{\infty} \frac{(1-\lambda\alpha)k^m - (-1)^{m-n}\alpha(1-\lambda)k^n}{1-\alpha} b_k \\
\le \; & 1 \, ,
\end{aligned}$$

since $0 \le \beta \le \alpha < 1$ and $f_m \in \overline{S}_H(m,n;\alpha;\lambda)$. Therefore $f_m * F_m \in \overline{S}_H(m,n;\alpha;\lambda) \subset \overline{S}_H(m,n;\beta;\lambda)$.

Now we show that the class $\overline{S}_H(m,n;\alpha;\lambda)$ is closed under convex combinations of its members.

Theorem 6. *The class* $\overline{S}_H(m,n;\alpha;\lambda)$ *is closed under convex combination.*

Proof. For $i = 1, 2, 3, ...$, let $f_{m_i} \in \overline{S}_H(m,n;\alpha;\lambda)$, where f_{m_i} is given by

$$f_{m_i}(z) = z - \sum_{k=2}^{\infty} a_{k_i} \, z^k + (-1)^{m-1} \sum_{k=1}^{\infty} b_{k_i} \, \overline{z}^k, \; (a_{k_i} \ge 0 \, ; \, b_{k_i} \ge 0 \, ; \;\; z \in U).$$

Then by Theorem 2, we have

$$\sum_{k=1}^{\infty} \left\{ \frac{(1-\lambda\alpha)k^m - \alpha(1-\lambda)k^n}{1-\alpha} \, a_{k_i} + \frac{(1-\lambda\alpha)k^m - (-1)^{m-n}\alpha(1-\lambda)k^n}{1-\alpha} \, b_{k_i} \right\} \le 2. \tag{4.4}$$

For $\sum\limits_{i=1}^{\infty} t_i = 1, 0 \leq t_i \leq 1$, the convex combination of f_{m_i} may be written as

$$\sum_{i=1}^{\infty} t_i\, f_{m_i}(z) = z - \sum_{k=2}^{\infty} (\sum_{i=1}^{\infty} t_i\, a_{k_i}) z^k + (-1)^{m-1} \sum_{k=1}^{\infty} (\sum_{i=1}^{\infty} t_i\, b_{k_i}) \overline{z}^k \ . \tag{4.5}$$

Then by (4.4), we have

$$\begin{aligned} & \sum_{k=1}^{\infty} \left\{ \frac{(1-\lambda\alpha)k^m - \alpha(1-\lambda)k^n}{1-\alpha} \sum_{i=1}^{\infty} t_i\, a_{k_i} + \right. \\ & \left. \frac{(1-\lambda\alpha)k^m - (-1)^{m-n}\alpha(1-\lambda)k^n}{1-\alpha} \sum_{i=1}^{\infty} t_i\, b_{k_i} \right\} \\ = & \sum_{i=1}^{\infty} t_i \left\{ \sum_{k=1}^{\infty} \left[\frac{(1-\lambda\alpha)k^m - \alpha(1-\lambda)k^n}{1-\alpha} \ a_{k_i} + \right.\right. \\ & \left.\left. \frac{(1-\lambda\alpha)k^m - (-1)^{m-n}\alpha(1-\lambda)k^n}{1-\alpha} \ b_{k_i} \right] \right\} \\ \leq & \ 2 \sum_{i=1}^{\infty} t_i = 2 \ . \end{aligned}$$

This is the condition required by (2.4) and so $\sum\limits_{i=1}^{\infty} t_i \ f_{m_i}(z) \in \overline{S}_H(m,n;\alpha;\lambda)$.

Theorem 7. *If* $f_m \in \overline{S}_H(m,n;\alpha;\lambda)$ *then* f_m *is convex in the disc*

$$|z| \leq \min_k \left\{ \frac{(1-\alpha)(1-b_1)}{k\left[1-\alpha-(\frac{(1-\lambda\alpha)-(-1)^{m-n}\alpha(1-\lambda)}{1-\alpha})\, b_1\right]} \right\}^{\frac{1}{k-1}} , \ k = 2,3,... \ .$$

Proof. Let $f_m \in \overline{S}_H(m,n;\alpha;\lambda)$, and let $0 < r < 1$, be fixed. Then $r^{-1} f_m(r\,z) \in \overline{S}_H(m,n;\alpha;\lambda)$ and we have

$$\sum_{k=1}^{\infty} k^2 (a_k + b_k) r^{k-1} = \sum_{k=2}^{\infty} k(a_k+b_k)(k\ r^{k-1})$$

$$\begin{aligned} \leq & \sum_{k=2}^{\infty} (\frac{(1-\lambda\alpha)k^m - \alpha(1-\lambda)k^n}{1-\alpha} a_k + \frac{(1-\lambda\alpha)k^m - (-1)^{m-n}\alpha(1-\lambda)k^n}{1-\alpha} b_k) k\, r^{k-1} \\ \leq & \ 1 - b_1 \end{aligned}$$

provided that

$$k\ r^{k-1} \leq \frac{1-b_1}{1-(\frac{(1-\lambda\alpha)k^m-(-1)^{m-n}\alpha(1-\lambda)k^n}{1-\alpha})\ b_1}$$

which is true if

$$k \leq \min_k \left\{ \frac{(1-\alpha)(1-b_1)}{k\left[1-\alpha-(\frac{(1-\lambda\alpha)-(-1)^{m-n}\alpha(1-\lambda)}{1-\alpha})\,b_1\right]} \right\}^{\frac{1}{k-1}},\ k=2,3,\ldots\ .$$

This complete the proof of Theorem 7.

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