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FINITE GROUPS WITH AT MOST FIVE NON T-SUBGROUPS

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Abstract. In this paper, we have characterized soluble groups by using the number of their non T-subgroups and also classified finite groups having exactly five non T-subgroups.

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1. Introduction

A group is said to be a T-group if every subnormal subgroup is normal. Thus the class of T-groups is just the class of all groups in which normality is a transitive relation. Finite groups whose all proper subgroups are T-groups have been studied in 1969 by Derek J. S. Robinson [3]. In that paper he proved that if all subgroups of a finite group are T-groups then G is soluble. Our aim is to study finite groups having non T-subgroups. In this paper we have characterized soluble groups by using the number of their non T-subgroups and also classified finite groups having exactly five non T-subgroups. We managed to prove that if all all subgroups of a finite group are T-groups except at most 4 subgroups then the group would be soluble. We also see that finite groups having more than 4 non T-subgroups are not soluble in general. Throughout this paper, simple group means non abelian simple group.

2. Main Results

We begin with the proof of the following lemma.

Lemma 2.1. *If G is a finite group all of whose proper subgroups are* T-*group except one, then G is soluble.*

Proof. Let H be a non T-subgroup of a finite group G. Then conjugate of H must be equal to itself (Since conjugate of a non T-group is a non T-group) and hence H is normal in G. Thus H and G/H are both soluble, by [3], and so G is soluble.

Theorem 2.2. *If G is a finite simple group such that G has exactly n non T-subgroups $(n \geq 1)$, then G is isomorphic to a subgroup of S_n.*

Proof. Let G be a finite simple group such that G has exactly n non T-subgroups. Let H_i, $i = 1, ..., n$, be the set of non T-subgroups. Since G is a T-group, each H_i is a proper subgroup of G. Also no H_i is a normal subgroup of G since G is simple. Now define a mapping $\alpha : G \longrightarrow Sym\{H_1, ..., H_n\}$ by $g\alpha : H_i \longrightarrow H_i^g$. Clearly α is a homomorphism. Since $Im(\alpha)$ is non trivial and G is simple, we must have $Ker(\alpha) = \{e\}$. So $\alpha : G \longrightarrow Sym\{H_1, ..., H_n\}$ is a faithful representation. But since $Sym\{H_1, ..., H_n\} \cong S_n$, there exists a faithful representation from G to $S_{n.}$ This implies that G is isomorphic to a subgroup of S_n.

Lemma 2.3. *Let G be a finite group with exactly n non T-subgroups and suppose that any finite group with exactly m non T-subgroups is soluble for $1 \leq m \leq n-1$. Then if G contains a normal non T-subgroup, G is soluble.*

Proof. Let N be a normal non T-subgroup of G. Clearly N contains less than n non T-subgroups and hence (by our assumption) N is soluble.
Now we prove that G/N is soluble. For this we prove that G/N contains less than n non T-subgroups. Let us suppose that G/N contains n non T-subgroups H_i/N for $i = 1, 2, \cdots, n$. Then clearly H_i's are the non T-subgroups of G different from N which shows that G contains more than n non T-subgroups, a contradiction. Thus G/N contains less than n non T-subgroups and hence G/N is soluble. This implies that G is soluble.

Lemma 2.4. *If G is a finite group all of whose proper subgroups are T-groups except for n, $n = 2, 3, 4$, then G is soluble.*

Proof. Let G be an insoluble group containing exactly n proper non T-subgroups, $L = \{H_1, \cdots, H_n\}$. Clearly each H_i contains fewer than n proper non T-subgroups and so is soluble by the Lemma 2.3, no H_i can be normal. By the Theorem 2.2 there is a homomorphism $\alpha : G \longrightarrow S_n$. Let K be the kernel of α. Then K cannot contain any of the H_i (since $K \leq \cap H_i$) and so K is soluble and in particular is a T-group. Then we must have G/K isomorphic to an insoluble subgroup of S_n. But S_n is soluble for $n = 2, 3, 4.$, a contradiction. Hence G is soluble for $n = 2, 3, 4$.

3. Classification of groups having exactly five non T-subgroups

Theorem 2.5. *Let G be a finite group having exactly five non* T*-subgroups. Then G is either soluble or G is isomorphic to one of A_5 and $SL(2, 5)$.*

Proof. Let G be an insoluble group containing exactly 5 proper non T-subgroups, $L = \{H_1, \cdots, H_5\}$. Clearly each H_i contains fewer than 5 proper non T-subgroups and so is soluble by Lemma 2.4, no H_i can be normal. By the Theorem 2.3 there is a homomorphism $\alpha : G \longrightarrow S_5$. Let K be the kernel of α. Then K cannot contain any of the H_i (since $K \leq \cap H_i$) and so K is soluble and in particular is a T-group. Then we must have G/K isomorphic to an insoluble subgroup of S_5. Since S_5 has too many non T-subgroups, we must have $G/K \cong A_5$. Since K is a soluble normal

subgroup with G/K simple, K is the soluble radical of G.
For some i let $S < H_i$, then S is a T-subgroup of G. We claim that $SK < H_i$. If $SK = H_i$, then $SK/K \cong S/S \cap K$. Since S is a T-subgroup of H_i. Therefore $S/S \cap K$ is a T-group and hence $H_i/K = H_iK/K$ is a T-group, a contradiction. Hence $SK < H_i$. Now we claim that $K \subset \Phi(H_i)$. If $K \nsubseteq \Phi(H_i)$, then there is a maximal subgroup M of H_i such that $K \nsubseteq M$. This implies $MK = H_i$, a contradiction. Therefore $K \subset \Phi(H_i)$ and hence, by Theorem 5.2.13(i) Robinson[2], $K \subset \Phi(G)$. This implies that K is nilpotent. Hence K is Dedekind. Now, by Theorem 9.3.5, Robinson [2], primes dividing $|\Phi(H_i)|$ also divide $|H_i/\Phi(H_i)|$ and hence $|H_i/K|$. This means that $|\Phi(H_i)|$ and hence $|K|$ is only divisible by 2 or 3.
Let $P_2(K)$ and $P_3(K)$ be the Sylow 2-subgroup and Sylow 3-subgroup of K. Then $K = P_2(K) \times P_3(K)$. Since $P_3(K)$ is a normal subgroup of odd order of K and K is Dedekind. Therefore $P_3(K)$ is Dedekind group of odd order and hence abelian. This implies that $K < C_G(P_3(K)) = C$. Also $P_3(K)$ is a characteristic subgroup of K and hence, by Theorem 1.5.6 (iii) Robinson [2], $P_3(K)$ is normal in G. This implies that $C_G(P_3(K)) = C$ is normal in G. That is $C \trianglelefteq G$. If $C \triangleleft G$ then $1 \neq C/K \triangleleft G/K$ but $G/K \cong A_5$, a contradiction. Therefore $C = G$. That is $C_G(P_3(K)) = G$ and so $P_3(K) \subseteq Z(G)$.
This implies that $P_3(K)$ is isomorphic to a subgroup of the Schur Multiplier of A_5. Since Schur Multiplier of A_5 has order 2 ($Theorem 12.3.2 of Karpilovsky$ [1]). Therefore $P_3(K) = \{e\}$. Thus we have $K = P_2(K)$.
Again we must have $P_2(G)$ (Sylow 2-subgroup of G) is a T-group. If it is abelian, then the same argument used for the Sylow 3-subgroup $P_3(K)$ shows K must be trivial. Hence a Sylow 2-subgroup $P_2(G)$ of G must be Hamiltonian group (the direct product of the quaternion group and an elementary abelian 2-group). We now have $P_2(G)/(P_2(G) \cap K)$ of order 4. Suppose that $P_2(G) \cap K \neq Z\left(P_2(G)\right)$. Then there is an element of $Z(P_2(G))$ not in K and hence $KC_G(K)$ is a normal subgroup of G properly containing K. Then $KC_G(K) = G$. Also $G/C_G(K) \cong K/\left(K \cap C_G\left(K\right)\right)$ is a 2-group and so is trivial. It follows that $K \leq C_G(K)$ and $C_G(K) = G$. Thus we must have $K \leq Z(P_2(G))$ and since K has index 4 in $P_2(G)$ we must have $K = Z(P_2(G))$. We now have $K = G' \cap Z(G)$ and so by Corollary 10.1.6 of Karpilovsky [1], K is isomorphic to a subgroup of the Schur Multiplier of A_5. Since Schur
Multiplier of A_5 has order 2 (Theorem 12.3.2 of Karpilovsky [1]), K has order 1 or 2. If $|K| = 1$, then $G \cong A_5$. If $|K| = 2$, then G is a representing group for A_5. Since representing groups for perfect groups are unique up to isomorphism (Corollary 11.5.8 of Karpilovsky [1]) and $SL(2,5)$ is a representing group for A_5 (*ie* $SL(2,5)/(SL(2,5)' \cap Z(SL(2,5))) \cong A_5$), we must have $G \cong SL(2,5)$.

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