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THE EFFECT OF CERTAIN INTEGRAL OPERATORS ON SOME CLASSES OF ANALYTIC FUNCTIONS

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Abstract. The aim of this paper is to introduced subclasses of Janowski functions with bounded boundary and bounded radius rotations of complex order b and of type ρ. And also to study the mapping properties of these classes under certain integral operators defined and studied by Breaz et. al recently.

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1. Introduction

Let A be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disc $E = \{z : |z| < 1\}$. Let $C_b(\rho)$ and $S_b^*(\rho)$ be the classes of convex and starlike functions of complex order b ($b \in \mathbb{C} - \{0\}$) and type ρ $(0 \leq \rho < 1)$ respectively studied by Frasin [5].
Let $P[A, B]$ be the class of functions $h(z)$, analytic in E with $h(0) = 1$ and

$$h(z) \prec \frac{1 + Az}{1 + Bz}, \; -1 \leq B < A \leq 1,$$

where the symbol $\prec$ stands for subordination. This class was introduced by Janowski [6]. It is noted that $P[1, -1] \equiv P$, where P is the well-known class of functions with positive real parts. Noor [9] generalized this concept of janowski functions and defined the class $P_k[A, B]$ as follows.
A function $p(z)$ is said to be in the class $P_k[A, B]$, if and only if,

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) h_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) h_2(z), \quad (1.1)$$

where $h_1(z), h_2(z) \in P[A,B]$. It is clear that $P_2[A,B] \equiv P[A,B]$ and $P_k[1,-1] \equiv P_k$, the well-known class given and studied by Pinchuk [13].
The important fact about the class $P_k[A,B]$ is that this class is convex set. That is, for $p_i(z) \in P_k[A,B]$ and $\alpha_i \in \mathbb{R}$ with $1 \leq i \leq n$, we have

$$\sum_{i=1}^{n} \alpha_i p_i(z) \in P_k[A,B]. \tag{1.2}$$

This can be easily seen from (1.1) by using the fact that the set $P[A,B]$ is convex [10]. By using all these concepts, we define the following classes.
A function $f(z) \in A$ is said to belong to the class $V_k[A,B,\rho,b]$, if and only if,

$$\frac{1}{1-\rho}\left[\left(1+\frac{1}{b}\frac{zf''(z)}{f'(z)}\right)-\rho\right] \in P_k[A,B],$$

where $-1 \leq B < A \leq 1$, $k \geq 2$, $0 \leq \rho < 1$ and $b \in \mathbb{C}-\{0\}$. When $\rho = 0$ and $b = 1$, we obtain the class $V_k[A,B]$ of janowski functions with bounded boundary rotation, first discussed by Noor [9].
Similarly, an analytic function $f(z) \in R_k[A,B,\rho,b]$, if and only if,

$$\frac{1}{1-\rho}\left[1+\frac{1}{b}\left(\frac{zf'(z)}{f(z)}-1\right)-\rho\right] \in P_k[A,B],$$

where $-1 \leq B < A \leq 1$, $k \geq 2$, $0 \leq \rho < 1$ and $b \in \mathbb{C}-\{0\}$. When $\rho = 0$ and $b = 1$, we obtain the class $R_k[A,B]$ of functions with bounded radius rotation, first discussed by Noor [9].
Let us consider the integral operators

$$F_n(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\alpha_1} \cdots \left(\frac{f_n(t)}{t}\right)^{\alpha_n} dt \tag{1.3}$$

and

$$F_{\alpha_1 \ldots \alpha_n}(z) = \int_0^z \left[f_1'(t)\right]^{\alpha_1} \ldots \left[f_n'(t)\right]^{\alpha_n} dt, \tag{1.4}$$

where $f_i(z) \in A$ and $\alpha_i > 0$ for all $i \in \{1,2,\ldots,n\}$.
These operators, given by (1.3) and (1.4), are introduced and studied by Breaz and Breaz [2] and Breaz et.al [4], respectively. Later on, Breaz and Güney [3] considered the above integral operators and they obtained their properties on the classes $C_b(\rho)$, $S_b^*(\rho)$ of convex and starlike functions of complex order b and type ρ introduced

and studied by Frasin [5]. Recently, Noor [11] discussed the effect of these integral opeartors on the classes $V_k(\rho, b)$ and $R_k(\rho, b)$.
In this paper, we investigate some propeties of the above integral operators $F_n(z)$ and $F_{\alpha_1 \dots \alpha_n}(z)$ for the classes $V_k[A, B, \rho, b]$ and $R_k[A, B, \rho, b]$ respectively.
In order to derive our main result, we need the following lemmas.

2. Preliminary Lemmas

Lemma 2.1. *Let β, γ, $A \in \mathbb{C}$ with $Re[\beta+\gamma] > 0$ and let $B \in [-1, 0]$ satisfy either*

$$Re\left[\beta\left[1 + \left(\left(1-\rho\right)A + \rho B\right)B\right] + \gamma\left(1 + B^2\right)\right] \geq \left|\left(\left(1-\rho\right)A + \rho B\right)\beta + \overline{\beta}B + B\left(\gamma + \overline{\gamma}\right)\right|,$$

when $B \in (-1, 0]$, or

$$Re\beta\left[1 + \left(1-\rho\right)A + \rho B\right] > 0 \text{ and } Re\left[\beta\left[1 - \left(\left(1-\rho\right)A + \rho B\right)\right] + 2\gamma\right] \geq 0,$$

when $B = -1$. If $h(z) = 1 + c_n z^n + c_{n+1} z^{n+1} \dots$ satisfies

$$\left\{h(z) + \frac{nzh'(z)}{\beta h(z) + \gamma}\right\} \prec \frac{1 + \left\{(1-\rho)A + \rho B\right\} z}{1 + Bz}, \tag{2.1}$$

then

$$h(z) \prec Q(z) \prec \frac{1 + \left\{(1-\rho)A + \rho B\right\} z}{1 + Bz}, \tag{2.2}$$

where

$$Q(z) = \frac{1}{\beta G(z)} - \frac{\gamma}{\beta},$$

and

$$G(z) = \begin{cases} \frac{1}{n}\int\limits_0^1 \left[\frac{1+Btz}{1+Bz}\right]^{\frac{\beta}{n}(1-\rho)\left(\frac{A}{B}-1\right)} t^{\frac{\beta+\gamma}{n}-1}dt, & B \neq 0, \\ \frac{1}{n}\int\limits_0^1 e^{\frac{\beta A}{n}(1-\rho)(t-1)z} t^{\frac{\beta+\gamma}{n}-1}dt, & B = 0. \end{cases}$$

From (2.2), we can deduce the sharp result that $h \in P(\beta)$, with

$$\beta = \beta(\rho, \beta, \gamma) = \min\ ReQ(z) = Q(-1).$$

This result is a special case of one, given in ([7], pp.109).
The following Lemma is a generalization of the result proved in [12].

Lemma 2.2. *Let $f(z) \in V_k[A, B, \rho]$. Then, $f(z) \in R_k[A, B, \beta]$, where*

$$\beta = \beta_1(\rho, 1, 0) = \frac{B\left[(1-\rho)\left(\frac{A}{B}-1\right)+1\right]}{(1-B)^{(1-\rho)\left(1-\frac{A}{B}\right)}-(1-B)}, \quad B \neq 0. \tag{2.3}$$

Proof. Let

$$\frac{zf'(z)}{f(z)} = p(z) = \left(\frac{k}{4}+\frac{1}{2}\right)h_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)h_2(z). \tag{2.4}$$

Logarithmic differentiation of (2.4) yields

$$\frac{(zf'(z))'}{f'(z)} = p(z) + \frac{zp'(z)}{p(z)}.$$

Since $f(z) \in V_k[A, B, \rho]$, it follows that

$$p(z) + \frac{zp'(z)}{p(z)} \in P_k[A, B, \rho]. \tag{2.5}$$

Now, we define

$$\phi(z) = \frac{1}{2}\left\{\frac{z}{(1-z)} + \frac{z}{(1-z)^2}\right\} = \frac{z\left(1-\frac{z}{2}\right)}{(1-z)^2}$$

and using (2.4) with convolution technique given by Noor [8], we have

$$\frac{\phi(z)}{z} * p(z) = \left(\frac{k}{4}+\frac{1}{2}\right)\left[\frac{\phi(z)}{z} * h_1(z)\right] - \left(\frac{k}{4}-\frac{1}{2}\right)\left[\frac{\phi(z)}{z} * h_2(z)\right],$$

which implies that

$$p(z) + \frac{zp'(z)}{p(z)} = \left(\frac{k}{4}+\frac{1}{2}\right)\left[h_1(z) + \frac{zh_1'(z)}{h_1(z)}\right] - \left(\frac{k}{4}-\frac{1}{2}\right)\left[h_2(z) + \frac{zh_2'(z)}{h_2(z)}\right]. \tag{2.6}$$

Thus, from (2.5) and (2.6), we have

$$h_i(z) + \frac{zh_i'(z)}{h_i(z)} \in P[A, B, \rho], \quad i = 1, 2.$$

We use Lemma 2.1 with $-1 \leq B < A \leq 1$, $n = 1$, $\gamma = 0$, $\beta = 1 > 0$, $\rho \in [0, 1)$ and $h = h_i$ in (2.1), to have $h_i \in P[A, B, \beta]$, where β is given in (2.3) and consequently $p(z) \in P_k[A, B, \beta]$, which gives the required result. This estimate is best possible, extremal function $Q(z)$ is given by

$$Q(z) = \begin{cases} \frac{(1+Bz)-(1+Bz)^{(1-\rho)\left(1-\frac{A}{B}\right)}}{Bz\left[(1-\rho)\left(\frac{A}{B}-1\right)+1\right]}, & \text{if } B \neq 0, \\ \\ \frac{1-e^{-(1-\rho)(Az)}}{(1-\rho)Az}, & \text{if } B = 0. \end{cases}$$

3. Main Results

Theorem 3.1. *Let $f_i(z) \in R_k[A, B, \rho, b]$ for $1 \leq i \leq n$ with $-1 \leq B < A \leq 1$, $0 \leq \rho < 1$, $b \in \mathbb{C} - \{0\}$. Also let $\alpha_i > 0$, $1 \leq i \leq n$. If*

$$\sum_{i=1}^{n} \alpha_i = 1,$$

then $F_n(z) \in V_k[A, B, \rho, b]$.

Proof. From (1.3), we have

$$\frac{zF_n''(z)}{F_n'(z)} = \sum_{i=1}^{n} \alpha_i \left(\frac{zf_i'(z)}{f_i(z)} - 1 \right). \tag{3.1}$$

By multiplying (3.1) with $\frac{1}{b}$, we have

$$\frac{1}{b} \frac{zF_n''(z)}{F_n'(z)} = \sum_{i=1}^{n} \alpha_i \frac{1}{b} \left(\frac{zf_i'(z)}{f_i(z)} - 1 \right)$$

or, equivalently

$$1 + \frac{1}{b} \frac{zF_n''(z)}{F_n'(z)} = \sum_{i=1}^{n} \alpha_i \left[1 + \frac{1}{b} \left(\frac{zf_i'(z)}{f_i(z)} - 1 \right) \right]. \tag{3.2}$$

Subtracting ρ from both sides of (3.2), we have

$$\left[\left(1 + \frac{1}{b} \frac{zF_n''(z)}{F_n'(z)} \right) - \rho \right] = \sum_{i=1}^{n} \alpha_i \left[\left(1 + \frac{1}{b} \left(\frac{zf_i'(z)}{f_i(z)} - 1 \right) \right) - \rho \right]. \tag{3.3}$$

Since $f_i(z) \in R_k[A, B, \rho, b]$ for $1 \leq i \leq n$, we have

$$\left[\left(1 + \frac{1}{b} \left(\frac{zf_i'(z)}{f_i(z)} - 1 \right) \right) - \rho \right] = (1 - \rho) \, p_i(z), \quad 1 \leq i \leq n, \tag{3.4}$$

where $p_i(z) \in P_k[A, B]$. Using (3.4) in (3.3), we obtain

$$\left[\left(1 + \frac{1}{b} \frac{zF_n''(z)}{F_n'(z)} \right) - \rho \right] = (1 - \rho) \sum_{i=1}^{n} \alpha_i p_i(z).$$

Using (1.2), we can have

$$\frac{1}{1 - \rho} \left[\left(1 + \frac{1}{b} \frac{zF_n''(z)}{F_n'(z)} \right) - \rho \right] \in P_k[A, B],$$

which implies that $F_n(z) \in V_k[A, B, \rho, b]$.

If we take $A = 1$, $B = -1$ in Theorem 3.1, we obtain the result proved in [11].

Corollary 3.2. *Let $f_i(z) \in R_k(\rho, b)$ for $1 \le i \le n$ with $0 \le \rho < 1$, $b \in \mathbb{C} - \{0\}$. Also let $\alpha_i > 0$, $1 \le i \le n$. If*

$$\sum_{i=1}^{n} \alpha_i = 1,$$

then $F_n(z) \in V_k(\rho, b)$.

If $k = 2$, $A = 1$, $B = -1$ in Theorem 3.1, we obtain the result proved in [3].

Corollary 3.3. *Let $f_i(z) \in S_b^*(\rho)$ for $1 \le i \le n$ with $0 \le \rho < 1$, $b \in \mathbb{C} - \{0\}$. Also let $\alpha_i > 0$, $1 \le i \le n$. If*

$$\sum_{i=1}^{n} \alpha_i = 1,$$

then $F_n(z) \in C_b(\rho)$.

Theorem 3.4. *Let $f_i(z) \in V_k[A, B, \rho, 1]$ for $1 \le i \le n$ with $-1 \le B < A \le 1$, $B \ne 0$, $0 \le \rho < 1$. Also let $\alpha_i > 0$, $1 \le i \le n$. If*

$$\sum_{i=1}^{n} \alpha_i = 1,$$

then $F_n(z) \in V_k[A, B, \beta, 1]$, where β is given by (2.3).

Proof. From (3.2) with $b = 1$, we have

$$\left(1 + \frac{zF_n''(z)}{F_n'(z)}\right) = \sum_{i=1}^{n} \alpha_i \left(\frac{zf_i'(z)}{f_i(z)}\right)$$

or, equivalently

$$\left[\left(1 + \frac{zF_n''(z)}{F_n'(z)}\right) - \beta\right] = \sum_{i=1}^{n} \alpha_i \left[\frac{zf_i'(z)}{f_i(z)} - \beta\right]. \tag{3.5}$$

Since $f_i(z) \in V_k[A, B, \rho, 1]$ for $1 \le i \le n$, then by using Lemma 2.2, we have $f_i(z) \in R_k[A, B, \beta, 1]$, where β is given by (2.3). That is,

$$\frac{zf_i'(z)}{f_i(z)} - \beta = (1 - \beta)\, p_i(z), \quad 1 \le i \le n, \tag{3.6}$$

where $p_i(z) \in P_k[A, B]$. Using (3.6) in (3.5), we obtain

$$\left[\left(1 + \frac{zF_n''(z)}{F_n'(z)}\right) - \beta\right] = (1 - \beta) \sum_{i=1}^{n} \alpha_i p_i(z).$$

Using (1.2), we can have

$$\frac{1}{1-\beta}\left[\left(1+\frac{zF_n''(z)}{F_n'(z)}\right)-\beta\right]\in P_k[A,B],$$

which implies that $F_n(z)\in V_k[A,B,\beta,1]$.

Set $n=1$ with $\alpha_1=1$, in Theorem 3.4, we obtain.

Corollary 3.5. *Let $f(z)\in V_k[A,B,\rho]$ for $-1\le B<A\le 1$, $B\neq 0$. Then the Alexandar operator $F_1(z)$, defined in [1], belongs to the class $V_k[A,B,\beta]$ for $-1\le B<A\le 1$, $B\neq 0$, where β is given by* (2.3) .

For $A=1, B=-1, \rho=0$ and $k=2$ in Corollary 3.5, we have the well known result, that is,

$$f(z)\in C\,(0)\Rightarrow F_1(z)\in C\left(\frac{1}{2}\right).$$

By setting $A=1$, $B=-1$ in Theorem 3.4, we obtain the following result.

Corollary 3.6. *Let $f_i(z)\in V_k\,(\rho,1)$ for $1\le i\le n$ with $0\le\rho<1$. Also let $\alpha_i>0$, $1\le i\le n$. If*

$$\sum_{i=1}^{n}\alpha_i=1,$$

then $F_n(z)\in V_k\,(\beta,1)$,where β is given by (2.3).

The above result in Corollary 3.6 is special case of the results proved in [11].

Theorem 3.7. *Let $f_i(z)\in V_k[A,B,\rho,b]$ for $1\le i\le n$ with $-1\le B<A\le 1$, $0\le\rho<1$, $b\in\mathbb{C}-\{0\}$. Also let $\alpha_i>0$, $1\le i\le n$. If*

$$\sum_{i=1}^{n}\alpha_i=1,$$

then $F_{\alpha_1\ldots\alpha_n}(z)\in V_k[A,B,\rho,b]$.

Proof. From (1.4), we have

$$\frac{F''_{\alpha_1\ldots\alpha_n}(z)}{F'_{\alpha_1\ldots\alpha_n}(z)}=\sum_{i=1}^{n}\alpha_i\left(\frac{f_i''(z)}{f_i'(z)}\right).$$

By multiplying both sides with $\dfrac{z}{b}$, we have

$$\frac{1}{b}\frac{zF''_{\alpha_1\ldots\alpha_n}(z)}{F'_{\alpha_1\ldots\alpha_n}(z)}=\sum_{i=1}^{n}\alpha_i\frac{1}{b}\left(\frac{zf_i''(z)}{f_i'(z)}\right)$$

This relation is equivalent to

$$\left[\left(1+\frac{1}{b}\frac{zF''_{\alpha_1\ldots\alpha_n}(z)}{F'_{\alpha_1\ldots\alpha_n}(z)}\right)-\rho\right]=\sum_{i=1}^{n}\alpha_i\left[\left(1+\frac{1}{b}\frac{zf''_i(z)}{f'_i(z)}\right)-\rho\right]. \tag{3.7}$$

Since $f_i(z)\in V_k[A,B,\rho,b]$ for $1\leq i\leq n$, we have

$$\left(1+\frac{1}{b}\frac{zf''_i(z)}{f'_i(z)}\right)-\rho=(1-\rho)\,p_i\left(z\right),\quad 1\leq i\leq n, \tag{3.8}$$

where $p_i\left(z\right)\in P_k[A,B]$. Using (3.8) in (3.7), we obtain

$$\left[\left(1+\frac{1}{b}\frac{zF''_{\alpha_1\ldots\alpha_n}(z)}{F'_{\alpha_1\ldots\alpha_n}(z)}\right)-\rho\right]=(1-\rho)\sum_{i=1}^{n}\alpha_i p_i\left(z\right).$$

Using the fact given in (1.2), we get

$$\frac{1}{1-\rho}\left[\left(1+\frac{1}{b}\frac{zF''_{\alpha_1\ldots\alpha_n}(z)}{F'_{\alpha_1\ldots\alpha_n}(z)}\right)-\rho\right]\in P_k[A,B].$$

This implies that $F_{\alpha_1\ldots\alpha_n}(z)\in V_k[A,B,\rho,b]$.

When $A=1$, $B=-1$ in Theorem 3.7, we obtain the result proved in [11].
Corollory 3.8. *Let $f_i(z)\in V_k\left(\rho,b\right)$ for $1\leq i\leq n$ with $0\leq\rho<1$, $b\in\mathbb{C}-\{0\}$. Also let $\alpha_i>0$, $1\leq i\leq n$. If*

$$\sum_{i=1}^{n}\alpha_i=1,$$

then $F_{\alpha_1\ldots\alpha_n}(z)\in V_k\left(\rho,b\right)$.

If $k=2$, $A=1$, $B=-1$ in Theorem 3.7, we have the result discussed in [3].
Corollory 3.9. *Let $f_i(z)\in S_b^*\left(\rho\right)$ for $1\leq i\leq n$ with $0\leq\rho<1$, $b\in\mathbb{C}-\{0\}$. Also let $\alpha_i>0$, $1\leq i\leq n$. If*

$$\sum_{i=1}^{n}\alpha_i=1,$$

then $F_{\alpha_1\ldots\alpha_n}(z)\in C_b\left(\rho\right)$.

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