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UNIVALENCE CRITERIONS FOR SOME INTEGRAL OPERATORS

Virgil Pescar, Daniel Breaz

Abstract. We consider the integral operators $D_{|\alpha|}$, $G_{|\alpha|,\gamma}$, $K_{\alpha_1,\alpha_2,\ldots,\alpha_n,|\gamma|,n}$ and for the functions $f \in \mathcal{A}$ we obtain sufficient conditions for univalence of these integral operators.

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1. Introduction

Let $\mathcal{A}$ be the class of functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

which are analityc in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of all univalent functions f in $\mathcal{U}$.

For $f \in \mathcal{A}$, the integral operator $D_{|\alpha|}$ is defined by

$$D_{|\alpha|}(z) = \left[|\alpha| \int_0^z u^{|\alpha|-1} f'(u)\, du \right]^{\frac{1}{|\alpha|}}, \tag{1.1}$$

for some complex numbers $\alpha\,(\alpha \neq 0)$.

Also, the integral operator $G_{|\alpha|,\gamma}$ for $f \in \mathcal{A}$ is given by

$$G_{|\alpha|,\gamma}(z) = \left[|\alpha| \int_0^z u^{|\alpha|-1} \left(\frac{h(u)}{u} \right)^{\gamma} du \right]^{\frac{1}{|\alpha|}}, \tag{1.2}$$

$\alpha,\ \gamma$ be complex numbers $(\alpha \neq 0)$.

The integral operator $K_{\alpha_1,\alpha_2,...,\alpha_n,|\gamma|,n}$ is defined by

$$K_{\alpha_1,\alpha_2,...,\alpha_n,|\gamma|,n}(z) = \left[|\gamma| \int_0^z u^{|\gamma|-1} \left(\frac{f_1(u)}{u} \right)^{\frac{1}{\alpha_1}} ... \left(\frac{f_n(u)}{u} \right)^{\frac{1}{\alpha_n}} du \right]^{\frac{1}{|\gamma|}} \tag{1.3}$$

for some complex numbers $\gamma, \alpha_1, ..., \alpha_n$, $\left(\gamma \neq 0;\ \alpha_j \neq 0; j = \overline{1,n}\right)$.

We need the following lemmas.

Lemma. 1.1. [2]. *Let* β *be a complex number,* $\mathrm{Re}\beta > 0$ *and* $f \in \mathcal{A}$.
If

$$\left| \frac{1-|z|^{2\beta}}{\beta} \right| \left| \frac{zf''(z)}{f'(z)} \right| \leq 1, \tag{1.4}$$

for all $z \in \mathcal{U}$, *then the function*

$$F_\beta(z) = \left[\beta \int_0^z u^{\beta-1} f'(u)\, du \right]^{\frac{1}{\beta}} \tag{1.5}$$

is regular and univalent in $\mathcal{U}$.

Lemma 1.2. (Schwarz [1]). *Let* f *be the function regular in the disk* $\mathcal{U}_R = \{z \in \mathbb{C} : |z| < R\}$ *with* $|f(z)| < M$, M *fixed. If* $f(z)$ *has in* $z = 0$ *one zero with multiply* $\geq m$, *then*

$$|f(z)| \leq \frac{M}{R^m} |z|^m,\ (z \in \mathcal{U}_R), \tag{1.6}$$

the equality (in the inequality (1.6) for $z \neq 0$*) can hold only if*

$$f(z) = e^{i\theta} \frac{M}{R^m} z^m,$$

where θ *is constant.*

2. Main results

Theorem 1. *Let* α *be a complex number,* $\mathrm{Re}\alpha > 0$ *and* $f \in \mathcal{A}$, $f(z) = z + a_2 z^2 + \ldots.$

If

$$\frac{1-|z|^{2\text{Re}\alpha}}{\text{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right|\leq 1, \tag{2.1}$$

for all $z\in\mathcal{U}$, *then the function*

$$D_{|\alpha|}(z)=\left[|\alpha|\int_0^z u^{|\alpha|-1}f'(u)\,du\right]^{\frac{1}{|\alpha|}} \tag{2.2}$$

is in the class $\mathcal{S}$.

Proof. Let us consider the function $\varphi:(0,\infty)\to\mathbb{R}$, $\varphi(x)=\frac{1-a^{2x}}{x}$, $0<a<1$. The function φ is decreasing and hence, since $|\alpha|\geq\text{Re}\alpha>0$, we obtain

$$\frac{1-|z|^{2|\alpha|}}{|\alpha|}\leq\frac{1-|z|^{2\text{Re}\alpha}}{\text{Re}\alpha},\ (z\in\mathcal{U}). \tag{2.3}$$

From (2.3) we have

$$\frac{1-|z|^{2|\alpha|}}{|\alpha|}\left|\frac{zf''(z)}{f'(z)}\right|\leq\frac{1-|z|^{2\text{Re}\alpha}}{\text{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right|, \tag{2.4}$$

for all $z\in\mathcal{U}$. Using (2.1) and (2.4) we get

$$\frac{1-|z|^{2|\alpha|}}{|\alpha|}\left|\frac{zf''(z)}{f'(z)}\right|\leq 1. \tag{2.5}$$

From (2.5) and by Lemma 1.1, for $\beta=|\alpha|$ it results that the function $D_{|\alpha|}$ is in the class $\mathcal{S}$.

Theorem 2. *Let* α,γ *be complex numbers,* $\text{Re}\alpha>0$, $\gamma\neq 0$ *and the function* $h\in\mathcal{A}$. *If*

$$\left|\frac{zh'(z)}{h(z)}-1\right|\leq\frac{(2\text{Re}\alpha+1)^{\frac{2\text{Re}\alpha+1}{2\text{Re}\alpha}}}{2|\gamma|}, \tag{2.6}$$

for all $z\in\mathcal{U}$, *then the function*

$$G_{|\alpha|,\gamma}(z)=\left[|\alpha|\int_0^z u^{|\alpha|-1}\left(\frac{h(u)}{u}\right)^{\gamma}du\right]^{\frac{1}{|\alpha|}} \tag{2.7}$$

is in the class $\mathcal{S}$.

Proof. Let us consider the function

$$p(z) = \int_0^z \left(\frac{h(u)}{u}\right)^{\gamma} du, \tag{2.8}$$

which is regular in $\mathcal{U}$.

We have

$$\frac{zp''(z)}{p'(z)} = \gamma\left[\frac{zh'(z)}{h(z)} - 1\right], (z \in \mathcal{U}). \tag{2.9}$$

The function $g(z) = \frac{zh'(z)}{h(z)} - 1$, $z \in \mathcal{U}$ is regular in $\mathcal{U}$, $g(0) = 0$ and from (2.6), by Lemma 1.2, we obtain

$$\left|\frac{zh'(z)}{h(z)} - 1\right| \le \frac{(2\mathrm{Re}\alpha + 1)^{\frac{2\mathrm{Re}\alpha+1}{2\mathrm{Re}\alpha}}}{2|\gamma|}|z|, \tag{2.10}$$

for all $z \in \mathcal{U}$.

From (2.9) and (2.10) we get

$$\frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}\left|\frac{zp''(z)}{p'(z)}\right| \le \frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}|z|\frac{(2\mathrm{Re}\alpha + 1)^{\frac{2\mathrm{Re}\alpha+1}{2\mathrm{Re}\alpha}}}{2|\gamma|}, \tag{2.11}$$

for all $z \in \mathcal{U}$.

Since

$$\max_{|z|<1}\left(\frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}|z|\right) = \frac{2}{(2\mathrm{Re}\alpha + 1)^{\frac{2\mathrm{Re}\alpha+1}{2\mathrm{Re}\alpha}}},$$

from (2.11), we have

$$\frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}\left|\frac{zp''(z)}{p'(z)}\right| \le 1, \quad (z \in \mathcal{U}). \tag{2.12}$$

Because

$$\frac{1-|z|^{2|\alpha|}}{|\alpha|}\left|\frac{zp''(z)}{p'(z)}\right| \le \frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}\left|\frac{zp''(z)}{p'(z)}\right|, \quad (z \in \mathcal{U}),$$

by (2.12) we obtain

$$\frac{1-|z|^{2|\alpha|}}{|\alpha|}\left|\frac{zp''(z)}{p'(z)}\right|\leq 1,\ (z\in\mathcal{U}). \tag{2.13}$$

From (2.8) we have $p'(z)=\left(\frac{h(z)}{z}\right)^{\gamma}$, $z\in\mathcal{U}$ and hence, by (2.13), Lemma 1.1, for $\beta=|\alpha|$, it result that $G_{|\alpha|,\gamma}\in\mathcal{S}$.

Theorem 3. *Let α_j, γ be complex numbers, $\alpha_j\neq 0$, $\mathrm{Re}\gamma>0$, $j=\overline{1,n}$, M_j positive real numbers and $f_j\in\mathcal{A}$, $f_j(z)=z+a_2z^2+...$, $j=\overline{1,n}$.*

If

$$\left|\frac{zf_j'(z)}{f_j(z)}-1\right|\leq M_j,\ \left(z\in\mathcal{U};\ \ j=\overline{1,n}\right) \tag{2.14}$$

and

$$\frac{M_1}{|\alpha_1|}+\frac{M_2}{|\alpha_2|}+...+\frac{M_n}{|\alpha_n|}\leq\frac{(2\mathrm{Re}\gamma+1)^{\frac{2\mathrm{Re}\gamma+1}{2\mathrm{Re}\gamma}}}{2}, \tag{2.15}$$

then the function

$$K_{\alpha_1,\alpha_2,...,\alpha_n,|\gamma|,n}(z)=\left[|\gamma|\int_0^z u^{|\gamma|-1}\left(\frac{f_1(u)}{u}\right)^{\frac{1}{\alpha_1}}...\left(\frac{f_n(u)}{u}\right)^{\frac{1}{\alpha_n}}du\right]^{\frac{1}{|\gamma|}} \tag{2.16}$$

is in the class $\mathcal{S}$.

Proof. We consider the function

$$g(z)=\int_0^z\left(\frac{f_1(u)}{u}\right)^{\frac{1}{\alpha_1}}...\left(\frac{f_n(u)}{u}\right)^{\frac{1}{\alpha_n}}du \tag{2.17}$$

and we observe that $g(0)=g'(0)-1=0$.

We have

$$p(z)=\frac{zg''(z)}{g'(z)}=\sum_{j=1}^{n}\left[\frac{1}{\alpha_j}\left(\frac{zf_j'(z)}{f_j(z)}-1\right)\right],\ (z\in\mathcal{U}). \tag{2.18}$$

From (2.14), (2.18), by Lemma 1.2., we obtain

$$\left|\frac{zg''(z)}{g'(z)}\right|\leq|z|\sum_{j=1}^{n}\frac{M_j}{|\alpha_j|},\ \ (z\in\mathcal{U}) \tag{2.19}$$

and hence, we get

$$\frac{1-|z|^{2\mathrm{Re}\gamma}}{\mathrm{Re}\gamma}\left|\frac{zg''(z)}{g'(z)}\right| \leq \frac{1-|z|^{2\mathrm{Re}\gamma}}{\mathrm{Re}\gamma}|z|\sum_{j=1}^{n}\frac{M_j}{|\alpha_j|}. \tag{2.19}$$

Since

$$\max_{|z|<1}\left(\frac{1-|z|^{2\mathrm{Re}\gamma}}{\mathrm{Re}\gamma}|z|\right) = \frac{2}{(2\mathrm{Re}\gamma+1)^{\frac{2\mathrm{Re}\gamma+1}{2\mathrm{Re}\gamma}}},$$

from (2.15) and (2.19) we obtain

$$\frac{1-|z|^{2\mathrm{Re}\gamma}}{\mathrm{Re}\gamma}\left|\frac{zg''(z)}{g'(z)}\right| \leq 1, \tag{2.20}$$

for all $z\in\mathcal{U}$.

We have

$$\frac{1-|z|^{2|\gamma|}}{|\gamma|}\left|\frac{zg''(z)}{g'(z)}\right| \leq \frac{1-|z|^{2\mathrm{Re}\gamma}}{\mathrm{Re}\gamma}\left|\frac{zg''(z)}{g'(z)}\right| \leq 1, \quad (z\in\mathcal{U}), \tag{2.21}$$

and by Lemma 1.1, we obtain that the function $K_{\alpha_1,\alpha_2\ldots,\alpha_n,|\gamma|,n}$ defined by (2.16) is in the class $\mathcal{S}$.

References

[1] O. Mayer, The Functions Theory of One Variable Complex, Bucureşti, 1981.

[2] N.N. Pascu, *An improvement of Becker's Univalence Criterion*, Proceedings of the Commemorative Session Simion Stoilow, University of Braşov, 1987, 43-48.

Virgil Pescar
Department of Mathematics
"Transilvania" University of Braşov
Faculty of Mathematics and Computer Science
500091 Braşov, Romania.
E-mail: *virgilpescar@unitbv.ro*

Daniel Breaz
Department of Mathematics
"1 Decembrie 1918" University
510009 Alba Iulia, Romania.
E-mail: *dbreaz@uab.ro*