Acta Universitatis Apulensis No. 25/2011
ISSN: 1582-5329 pp. 133-144

FIRST ORDER LINEAR DIFFERENTIAL SUBORDINATIONS FOR A GENERALIZED OPERATOR

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Abstract. In this work, we obtain some results for first order linear strong differential subordination for new subclass of generalized operator.

2000 *Mathematics Subject Classification*: 30C45.

1. Introduction And Preliminaries

Let $H = H(U)$ denote the class of functions analytic in U. For n, a positive integer and $a \in C$, and let

$$H[a,n] = \{f \in H; f(z) = a + a_n z^n + \ldots, z \in U\}.$$

Let A be the class of functions f of the form

$$f(z) = z + a_2 z^2 + \ldots, z \in U$$

which are analytic in the unit disk. Let S^* and C be the class of starlike and convex functions respectively given by

$$S^* = \{f \in A, \Re\frac{zf'(z)}{f(z)} > 0\}$$

and

$$C = \{f \in A, \Re\{1 + \frac{zf''(z)}{f'(z)}\} > 0\}.$$

For $f \in A$, Ruscheweyh [6] considered the following generalized integral operator

$$f(z) = I_{\delta,\gamma}(F)(z) = [\frac{\delta+\gamma}{z^\gamma}\int_0^z t^{\gamma-1}F^\delta(t)\,dt]^{\frac{1}{\delta}}, (\delta > 0, \ \Re\,\gamma > 0).$$

We may also see some other results for other types of integral operators studied by different authors, namely Pescar and Breaz [9], El-Ashwah and Aouf[10], few to mention.

Let

$$D^k_{\alpha,\beta,\lambda,\mu}f(z) = \sum_{n=2}^{\infty}[\beta(n-1)(\lambda-\alpha)+1]^k G(\mu,n)\, a_n z^n$$

where $z \in U$ for $\alpha \geq 0,\ \ \beta \geq 0,\ \ \lambda \geq 0$ and $k \in N_0 = N \cup \{0\},\ \ \mu \geq 0$.

Remark 1.3.

1. When $\alpha = 0,\ \ \beta = 1$ and $\lambda = 1$,$\mu = 0$, we get Sălăgean differential operator (see [8]).
2. When $k = 0$ gives Ruscheweyh (see [7]).
3. When $\alpha = 0,\ \ \beta = 1$ and $\lambda = 1$ and $\mu = 0$, we get Al-Oboudi differential operator (see [3]).
4. When $\mu = 0$, the operator reduces to Darus and Ibrahim (see [2]).

Definition 1.4.*Let $f \in A$, then $f \in S^k_{\alpha,\beta,\lambda,\mu}$ if and only if for $z \in U$:*

$$\Re\{\frac{z(D^k_{\alpha,\beta,\lambda,\mu}f(z))'}{D^k_{\alpha,\beta,\lambda,\mu}f(z)}\} > 0, .$$

In our present investigation of the first order linear strong differential subordination for new subclass of generalized operator, we need the following definitions and lemmas:

Definition 1.5. [1]*Let $H(z,\xi)$ be analytic in $U \times \bar{U}$ and let f analytic and univalent in U. The function $H(z,\xi)$ is strongly subordinate to f, written $H(z,\xi) \prec\prec\ f(z)$ if for each $\xi \in \bar{U}$, the function of z,$H(z,\xi)$ is subordinate to f.*

Remark 1.6. Since f is analytic and univalent, Definition 1.4 is equivalent to $H(0,\xi) = f(0)$ and $H(U \times \bar{U}) \subset\ f(U)$.

Definition 1.7. [4]*We denote by Q the set of functions f that are analytic and injective in $\bar{U}\backslash E(f)$,where*

$$E(f) = \{\zeta \in \partial U; \lim_{z\to\zeta} f(z) = \infty\}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U\backslash E(f)$. The subclass of Q for which $f(0) = a$ is denoted by $Q(a)$.

Definition 1.8. [5] *Let Ω be a set in C, $q \in Q$ and n be a positive integer. The class of admissible functions $\psi_n[\Omega, q]$ consists of those functions $\psi : C^2 \times U \times \bar{U}$ that satisfy the admissibility condition:*

$$\psi(r, s; z, \xi) \notin \Omega$$

whenever $r = q(\zeta)$, $s = m\zeta q'(\zeta)$, $z \in U, \xi \in \bar{U}$, $\zeta \in \partial U \backslash E(f)$ and $m \geq n$.

Lemma 1.9. [4] *Let $q \in Q(a)$, with $q(0) = a$ and let $p(z) = a + a_n z^n + \cdots$ be analytic in U, with $p(z) \neq a$ and $n \geq 1$. If p is not subordinate to q, then there exist points $z_0 = r_0 e^{i\theta_0} \in U$ and $\zeta_0 \in \partial U \backslash E(q)$, and an $m \geq n \geq 1$*

1. $p(z_0) = q(\zeta_0)$
2. $z_0 p'(z_0) = m\zeta_0 q(\zeta_0)$.

2.Main Results

By referring to Definition 1.5, we state the first order linear strong differential subordination for new subclass of generalized operator as follows:

Definition 2.1. *A strong differential subordination for some subclasses of generalized operator of the form*

$$A(z,\xi)[\frac{z^2(D^k_{\alpha,\beta,\lambda,\mu}f(z))''}{D^k_{\alpha,\beta,\lambda,\mu}f(z)} + \frac{z(D^k_{\alpha,\beta,\lambda,\mu}f(z))'}{D^k_{\alpha,\beta,\lambda,\mu}f(z)} - \frac{z^2(D^k_{\alpha,\beta,\lambda,\mu}f(z))'^2}{(D^k_{\alpha,\beta,\lambda,\mu}f(z)\,)^2}]+$$

$$B(z,\xi)\frac{z(D^k_{\alpha,\beta,\lambda,\mu}f(z))'}{D^k_{\alpha,\beta,\lambda,\mu}f(z)} \prec\prec h(z), \ \ z \in U, \ \xi \in \bar{U} \tag{1}$$

where $p(z) = \frac{z(D^k_{\alpha,\beta,\lambda,\mu}f(z))'}{D^k_{\alpha,\beta,\lambda,\mu}f(z)}$ and $\psi(r, s; z, \xi) = A(z, \xi)zp'(z) + B(z, \xi)p(z)$ is analytic in U for all $\xi \in \bar{U}$ and $h(z)$ is analytic in U is called first order linear strong differential subordination for new subclass of generalized operator.

Remark 2.2. If $A(z, \xi) = 1$, $B(z, \xi) = 0$ and $h(z)$ is convex function then (1) becomes

$$\frac{z^2(D^k_{\alpha,\beta,\lambda,\mu}f(z))''}{D^k_{\alpha,\beta,\lambda,\mu}f(z)} + \frac{z(D^k_{\alpha,\beta,\lambda,\mu}f(z))'}{D^k_{\alpha,\beta,\lambda,\mu}f(z)}$$

$$-\frac{z^2(D^k_{\alpha,\beta,\lambda,\mu}f(z))'^2}{(D^k_{\alpha,\beta,\lambda,\mu}f(z))^2} \prec\ h(z), \quad z \in U.$$

Theorem 2.3. *Let* $\frac{z(D^k_{\alpha,\beta,\lambda,\mu}f(z))'}{D^k_{\alpha,\beta,\lambda,\mu}f(z)} \in H[0,n]$, $A : U \times \bar{U} \to\ C$, $B : U \times \bar{U} \to\ C$ *with* $\psi(r,s;z,\xi)$ *analytic in* U *for all* $,\xi \in \bar{U}$ *and*

$$\Re[nA(z,\xi) + B(z,\xi)] \geq 1, \ \ \Re A(z,\xi) \geq 0.$$

If

$$A(z,\xi)[\frac{z^2(D^k_{\alpha,\beta,\lambda,\mu}f(z))''}{D^k_{\alpha,\beta,\lambda,\mu}f(z)} + \frac{z(D^k_{\alpha,\beta,\lambda,\mu}f(z))'}{D^k_{\alpha,\beta,\lambda,\mu}f(z)}$$

$$-\frac{z^2(D^k_{\alpha,\beta,\lambda,\mu}f(z))'^2}{(D^k_{\alpha,\beta,\lambda,\mu}f(z))^2}] + B(z,\xi)\frac{z(D^k_{\alpha,\beta,\lambda,\mu}f(z))'}{D^k_{\alpha,\beta,\lambda,\mu}f(z)} \ \prec\prec\ (n^2+1)M\,z\,, \ \ z \in U, \ \xi \in \bar{U}, \ M > 0 \quad (2)$$

then

$$\frac{z(D^k_{\alpha,\beta,\lambda,\mu}f(z))'}{D^k_{\alpha,\beta,\lambda,\mu}f(z)} \prec Mz.$$

Proof. Let $\psi : C^2 \times U \times \bar{U} \to\ C$

$$\psi(r,s;z,\xi) = A(z,\xi)[\frac{z^2(D^k_{\alpha,\beta,\lambda,\mu}f(z))''}{D^k_{\alpha,\beta,\lambda,\mu}f(z)} + \frac{z(D^k_{\alpha,\beta,\lambda,\mu}f(z))'}{D^k_{\alpha,\beta,\lambda,\mu}f(z)}$$

$$-\frac{z^2(D^k_{\alpha,\beta,\lambda,\mu}f(z))'^2}{(D^k_{\alpha,\beta,\lambda,\mu}f(z))^2}] + B(z,\xi)\frac{z(D^k_{\alpha,\beta,\lambda,\mu}f(z))'}{D^k_{\alpha,\beta,\lambda,\mu}f(z)},$$

and (2) becomes

$$\psi(r,s;z,\xi) \ \prec\prec\ (n^2+1)Mz \quad (3)$$

since $h(z) = (n^2+1)M\,z$, it gives $h(z) = U(0,(n^2+1)M\,)$. In this case (3) is equivalent to

$$\psi(r, s; z, \xi) \in U(0, (n^2 + 1)M\ ). \tag{4}$$

Suppose that

$$\frac{z(D^k_{\alpha,\beta,\lambda,\mu} f(z))'}{D^k_{\alpha,\beta,\lambda,\mu} f(z)}$$

is not subordinated to $q(z) = M\,z$. Then by using Lemma 1.9, we have that there exist $z_0 \in U$ and $\zeta_0 \in \partial U$ such that

$$\frac{z_0(D^k_{\alpha,\beta,\lambda,\mu} f(z_0))'}{D^k_{\alpha,\beta,\lambda,\mu} f(z_0)} = q(z_0) = Me^{i\theta_0},$$

where $\theta_0 \in R$ when $|\zeta_0| = 1$ and

$$\frac{z_0^2(D^k_{\alpha,\beta,\lambda,\mu} f(z_0))''}{D^k_{\alpha,\beta,\lambda,\mu} f(z_0)} + \frac{z_0(D^k_{\alpha,\beta,\lambda,\mu} f(z_0))'}{D^k_{\alpha,\beta,\lambda,\mu} f(z_0)}$$

$$-\frac{z_0^2(D^k_{\alpha,\beta,\lambda,\mu} f(z_0))'^2}{(D^k_{\alpha,\beta,\lambda,\mu} f(z_0))^2} = m\ \zeta_0\, h'(\zeta_0) = K\, e^{i\theta_0}, \quad K \geq\ \ nM\,.$$

Hence we obtain

$$|\psi(r, s; z_0, \xi)| \ = \ \left|A(z_0, \xi)\ K\, e^{i\theta_0} + B(z_0, \xi)\, M\, e^{i\theta_0}\right|$$

$$= \quad |A(z_0, \xi)\ K\ + B(z_0, \xi)\, M\,|$$

$$\geq \quad \Re\ |A(z_0, \xi)\ K\ + B(z_0, \xi)\, M\,|\ \geq\ K\,\Re A(z_0, \xi)\ + M\Re B(z_0, \xi)$$

$$\geq\ M[\ n\,\Re A(z_0, \xi)\ + \Re B(z_0, \xi)\,] \geq\ M,$$

Since this result contradicts (4), we conclude that the assumption made concerning the subordination relation between p and q is false, hence

$$\frac{z(D^k_{\alpha,\beta,\lambda,\mu} f(z))'}{D^k_{\alpha,\beta,\lambda,\mu} f(z)} \prec\ M\,z,\ \ z \in U.$$

When $\phi(z) = 1$ in Theorem 2.3 we get:

Corollary 2.4. *Let* $\frac{z(D^k_{\alpha,\beta,\lambda,\mu}f(z))'}{D^k_{\alpha,\beta,\lambda,\mu}f(z)} \in H[0,n]$, $A : U \times \bar{U} \to C$, $B : U \times \bar{U} \to C$ *with*

$$A(z,\xi)[\frac{z^2(D^k_{\alpha,\beta,\lambda,\mu}f(z))''}{D^k_{\alpha,\beta,\lambda,\mu}f(z)} + \frac{z(D^k_{\alpha,\beta,\lambda,\mu}f(z))'}{D^k_{\alpha,\beta,\lambda,\mu}f(z)} - \frac{z^2(D^k_{\alpha,\beta,\lambda,\mu}f(z))'^2}{(D^k_{\alpha,\beta,\lambda,\mu}f(z))^2}] + B(z,\xi)\frac{z(D^k_{\alpha,\beta,\lambda,\mu}f(z))'}{D^k_{\alpha,\beta,\lambda,\mu}f(z)},$$

analytic function in U *for all* $,\xi \in \bar{U}$ *and*

$$\Re[nA(z,\xi) + B(z,\xi)] \geq 1, \ \Re A(z,\xi) \geq 0.$$

If

$$A(z,\xi)[\frac{z^2(D^k_{\alpha,\beta,\lambda,\mu}f(z))''}{D^k_{\alpha,\beta,\lambda,\mu}f(z)} + \frac{z(D^k_{\alpha,\beta,\lambda,\mu}f(z))'}{D^k_{\alpha,\beta,\lambda,\mu}f(z)} - \frac{z^2(D^k_{\alpha,\beta,\lambda,\mu}f(z))'^2}{(D^k_{\alpha,\beta,\lambda,\mu}f(z))^2}]$$

$$+B(z,\xi)\frac{z(D^k_{\alpha,\beta,\lambda,\mu}f(z))'}{D^k_{\alpha,\beta,\lambda,\mu}f(z)} \prec\prec (n^2+1)M\,z\,, \ \ z \in U, \ \xi \in \bar{U}, \ M > 0 \tag{5}$$

then

$$\frac{z(D^k_{\alpha,\beta,\lambda,\mu}f(z))'}{D^k_{\alpha,\beta,\lambda,\mu}f(z)} \prec Mz.$$

For $k = 0$, we get the following corollary.

Corollary 2.5. *Let* $\frac{z\,f'(z)}{f(z)} \in H[0,n]$ *and* $A : U \times \bar{U} \to C$, $B : U \times \bar{U} \to C$ *with*

$$A(z,\xi)[\frac{z^2 f''(z)}{f(z)} + \frac{z\,f'(z)}{f(z)} - \frac{z^2(f'(z))^2}{(f(z))^2}] + B(z,\xi)\frac{zf'(z)}{f(z)}\,,$$

analytic function in U *for all* $,\xi \in \bar{U}$ *and*

$$\Re[nA(z,\xi) + B(z,\xi)] \geq 1, \ \Re A(z,\xi) \geq 0.$$

If

$$A(z,\xi)[\frac{z^2 f''(z)}{f(z)} + \frac{z\,f'(z)}{f(z)} - \frac{z^2(f'(z))^2}{(f(z))^2}]$$

$$+B(z,\xi)\frac{zf'(z)}{f(z)} \prec\prec (n^2+1)\,M\ z,\ z\in U,\ ,\xi\in\bar{U}, M>0, \tag{6}$$

then

$$\frac{z\,f'(z)}{f(z)} \prec M\ z.$$

Theorem 2.6. *Let* $\frac{z(D^k_{\alpha,\beta,\lambda,\mu}f(z))'}{D^k_{\alpha,\beta,\lambda,\mu}f(z)} \in H[1,n]$ *and* $A: U\times\bar{U}\to C,\ B: U\times\bar{U}\to C$ *with* $\psi(r,s;z,\xi)$ *a function of* z*, analytic in for any* $\xi\in\bar{U}$ *and*

$$\Re A(z,\xi)\geq 0,\quad \Im B(z,\xi)\ \leq n\,\Re A(z,\xi).$$

If

$$\Re\{A(z,\xi)[\frac{z^2(D^k_{\alpha,\beta,\lambda,\mu}f(z))''}{D^k_{\alpha,\beta,\lambda,\mu}f(z)}+\frac{z(D^k_{\alpha,\beta,\lambda,\mu}f(z))'}{D^k_{\alpha,\beta,\lambda,\mu}f(z)}$$

$$-\frac{z^2(D^k_{\alpha,\beta,\lambda,\mu}f(z))'^2}{(D^k_{\alpha,\beta,\lambda,\mu}f(z))^2}]+B(z,\xi)\frac{z(D^k_{\alpha,\beta,\lambda,\mu}f(z))'}{D^k_{\alpha,\beta,\lambda,\mu}f(z)}\}>0 \tag{7}$$

then

$$\Re\frac{z(D^k_{\alpha,\beta,\lambda,\mu}f(z))'}{D^k_{\alpha,\beta,\lambda,\mu}f(z)}\ >0, z\in U.$$

Proof. Let $\psi: C^2\times U\times\bar{U}\to\ C$,

$$\psi(r,s;z,\xi)=A(z,\xi)[\frac{z^2(D^k_{\alpha,\beta,\lambda,\mu}f(z))''}{D^k_{\alpha,\beta,\lambda,\mu}f(z)}+\frac{z(D^k_{\alpha,\beta,\lambda,\mu}f(z))'}{D^k_{\alpha,\beta,\lambda,\mu}f(z)}$$

$$-\frac{z^2(D^k_{\alpha,\beta,\lambda,\mu}f(z))'^2}{(D^k_{\alpha,\beta,\lambda,\mu}f(z))^2}]+B(z,\xi)\frac{z(D^k_{\alpha,\beta,\lambda,\mu}f(z))'}{D^k_{\alpha,\beta,\lambda,\mu}f(z)},$$

In this case, (7) becomes

$$\Re\psi(r,s;z,\xi)\ >0,\ \ z\in U,\quad \xi\in\bar{U}. \tag{8}$$

Since $h(z) = \frac{1+Az}{1+Bz}, \quad -1 \le B < A \le 1$, and

$$h(U) = \{\omega \in C : \Re\omega(z) > 0\}$$

from which we have that (8) becomes

$$\psi(r, s; z, \xi) \prec \frac{1+Az}{1+Bz}, \quad -1 \le B < A \le 1.$$

Suppose $\Re\{\frac{z(D^k_{\alpha,\beta,\lambda,\mu}f(z))'}{D^k_{\alpha,\beta,\lambda,\mu}f(z)}\} < 0$, meaning $p(z) = \frac{z(D^k_{\alpha,\beta,\lambda,\mu}f(z))'}{D^k_{\alpha,\beta,\lambda,\mu}f(z)}$ is not subordinate to $h(z) = \frac{1+Az}{1+Bz}, \quad -1 \le B < A \le 1$. Using Lemma 1.5, we have that there exist $z_0 \in U$ and $\zeta_0 \in \partial U$ with $|\zeta_0| = 1$ such that $\frac{z_0(D^k_{\alpha,\beta,\lambda,\mu}f(z_0))'}{\phi(z_0)D^k_{\alpha,\beta,\lambda,\mu}f(z_0)} = h(\zeta_0) = \rho i$, and

$$\frac{z_0^2(D^k_{\alpha,\beta,\lambda,\mu}f(z_0))''}{D^k_{\alpha,\beta,\lambda,\mu}f(z_0)} + \frac{z_0(D^k_{\alpha,\beta,\lambda,\mu}f(z_0))'}{D^k_{\alpha,\beta,\lambda,\mu}f(z_0)}$$

$$-\frac{z_0^2(D^k_{\alpha,\beta,\lambda,\mu}f(z_0))'^2}{(D^k_{\alpha,\beta,\lambda,\mu}f(z_0))^2} = m\ \zeta_0\, h'(\zeta_0) = \sigma,$$

where $\rho, \sigma \in R$ and $\sigma \le -n(1+\rho^2),\ n \ge 1$. Then we obtain:

$$\Re\psi(r, s; z_0, \xi) = \Re\psi(\rho i, \sigma; z_0, \xi)$$

$$= \Re[A(z_0, \xi)\,\sigma + B(z_0, \xi)\rho i] = \Re\{A(z_0, \xi)\,\sigma + [B(z_0, \xi) + i\,B(z_0, \xi)]\rho i\}$$

$$= \sigma\,\Re A(z_0, \xi) - \rho\Im B(z_0, \xi) \le -n(1+\rho^2)\Re A(z_0, \xi) - \rho\Im B(z_0, \xi)$$

$$\le -\frac{n}{2}\rho^2\Re A(z_0, \xi) - \rho\Im B(z_0, \xi) - \frac{n}{2} \le 0.$$

Hence $\Re\psi(r, s; z_0, \xi) \le 0$ which contradicts (2.8) and we conclude that

$$\frac{z(D^k_{\alpha,\beta,\lambda,\mu}f(z))'}{D^k_{\alpha,\beta,\lambda,\mu}f(z)} > 0,\ z \in U.$$

When $k = 0$, we get

Corollary 2.7. *Let* $\frac{zf'(z)}{f(z)} \in H[1,n]$ *and* $A : U \times \bar{U} \to C$, $B : U \times \bar{U} \to C$ *with* $\psi(r,s;z,\xi)$ *a function of* z, *analytic in for any* $\xi \in \bar{U}$ *and*

$$\Re A(z,\xi) \geq 0, \quad \Im B(z,\xi) \leq n\,\Re A(z,\xi).$$

If

$$A(z,\xi)[\frac{z^2 f''(z)}{f(z)} + \frac{z\,f'(z)}{f(z)} - \frac{z^2(f'(z))^2}{(f(z))^2}] + B(z,\xi)\frac{zf'(z)}{f(z)} > 0$$

then

$$\frac{zf'(z)}{f(z)} > 0, z \in U.$$

Theorem 2.8. *Let* $\frac{z(D^k_{\alpha,\beta,\lambda,\mu}f(z))'}{D^k_{\alpha,\beta,\lambda,\mu}f(z)} \in H[0,n]$, $A : U \times \bar{U} \to C$, $B : U \times \bar{U} \to C$ *with* $\psi(r,s;z,\xi)$ *analytic in* U *for all* $\xi \in \bar{U}$ *and*

$$\Re\,A(z,\xi) \geq 0, \Im B(z,\xi) \leq n\Re\,A(z,\xi)[-n\Re\,A(z,\xi) + z].$$

If

$$A(z,\xi)[\frac{z^2(D^k_{\alpha,\beta,\lambda,\mu}f(z))''}{D^k_{\alpha,\beta,\lambda,\mu}f(z)} + \frac{z(D^k_{\alpha,\beta,\lambda,\mu}f(z))'}{D^k_{\alpha,\beta,\lambda,\mu}f(z)} \tag{9}$$

$$-\frac{z^2(D^k_{\alpha,\beta,\lambda,\mu}f(z))'^2}{(D^k_{\alpha,\beta,\lambda,\mu}f(z))^2}] + B(z,\xi)\frac{z(D^k_{\alpha,\beta,\lambda,\mu}f(z))'}{D^k_{\alpha,\beta,\lambda,\mu}f(z)} \prec\prec M\ z,\ z \in U, ,\xi \in \bar{U}, M > 0$$

then

$$\frac{z(D^k_{\alpha,\beta,\lambda,\mu}f(z))'}{D^k_{\alpha,\beta,\lambda,\mu}f(z)} \prec \frac{1+Az}{1+Bz}, \quad -1 \leq B < A \leq 1,\ z \in U.$$

Proof. Let $\psi : C^2 \times U \times \bar{U} \to C$, and

$$\psi(r,s;z,\xi) = A(z,\xi)[\frac{z^2(D^k_{\alpha,\beta,\lambda,\mu}f(z))''}{D^k_{\alpha,\beta,\lambda,\mu}f(z)} + \frac{z(D^k_{\alpha,\beta,\lambda,\mu}f(z))'}{D^k_{\alpha,\beta,\lambda,\mu}f(z)}$$

$$-\frac{z^2(D^k_{\alpha,\beta,\lambda,\mu}f(z))'^2}{(D^k_{\alpha,\beta,\lambda,\mu}f(z))^2}] + B(z,\xi)\frac{z(D^k_{\alpha,\beta,\lambda,\mu}f(z))'}{D^k_{\alpha,\beta,\lambda,\mu}f(z)},$$

then (9) becomes

$$\psi(r,s;z,\xi) \prec\prec Mz \tag{10}$$

since $h(z) = M\,z$, we have $h(z) = U(0, M\,)$. Thus

$$\psi(r,s;z,\xi) \subset U, z \in U, ,\xi \in \bar{U}. \tag{11}$$

Suppose that $\frac{z(D^k_{\alpha,\beta,\lambda,\mu}f(z))'}{D^k_{\alpha,\beta,\lambda,\mu}f(z)}$ is not subordinated to $q(z) = \frac{1+Az}{1+Bz}$, $-1 \le B < A \le 1$. then ,by using Lemma 1.9, we have that there exist $z_0 \in U$ and $\zeta_0 \in \partial U$ such that

$$\frac{z(D^k_{\alpha,\beta,\lambda,\mu}f(z))'}{D^k_{\alpha,\beta,\lambda,\mu}f(z)} = q(\zeta_0) = \upsilon i,$$

$$\frac{z^2(D^k_{\alpha,\beta,\lambda,\mu}f(z))''}{D^k_{\alpha,\beta,\lambda,\mu}f(z)} + \frac{z(D^k_{\alpha,\beta,\lambda,\mu}f(z))'}{D^k_{\alpha,\beta,\lambda,\mu}f(z)}$$

$$-\frac{z^2(D^k_{\alpha,\beta,\lambda,\mu}f(z))'^2}{(D^k_{\alpha,\beta,\lambda,\mu}f(z))^2}] = m\zeta_0 q'(\zeta_0) = \rho$$

where $\upsilon, \rho \in R$ and $\rho \le \frac{n}{2}(1+\upsilon^2)$, $n \ge 1$. Then we obtain

$$\Re\,\psi(r,s;z_0,\xi) = \Re\,\psi(\upsilon i,\ \rho; z_0,\xi)$$

$$= \Re\,[A(z_0,\xi)\,\rho + B(z_0,\xi)\,\upsilon i] = \rho\Re\,A(z_0,\xi) - \upsilon\ \Im B(z_0,\xi)$$

$$\le -\frac{n}{2}(1+\upsilon^2)\Re\,A(z_0,\xi) - \upsilon\ \Im B(z_0,\xi)$$

$$\le\ -\frac{n}{2}\Re\,A(z_0,\xi) - \upsilon\ \Im B(z_0,\xi) -\ \frac{n}{2}\upsilon\Re\,A(z_0,\xi) \le -1.$$

Hence, we have

$$\Re\,\psi(r,s;z_0,\xi) \le -1,$$

which contradicts (11) and we conclude that

$$\frac{z(D^k_{\alpha,\beta,\lambda,\mu} f(z))'}{D^k_{\alpha,\beta,\lambda,\mu} f(z)} \prec \frac{1+Az}{1+Bz}, \quad -1 \leq B < A \leq 1,\ z \in U.$$

When $k = 0$, $M = 1$ and $q(z) = \frac{1+z}{1-z}$ in Theorem 2.8 we get:

Corollary 2.9. *Let* $\frac{z\, f'(z)}{f(z)} \in H[1, n]$ *and*

$$A(z,\xi)\,[\frac{z^2 f''(z)}{f(z)} + \frac{z\, f'(z)}{f(z)} - \frac{z^2 (f'(z))^2}{(f(z))^2}] + B(z,\xi)\frac{z f'(z)}{f(z)}$$

analytic function in U *for all* $\xi \in \bar{U}$ *and*

$$\Re\, A(z,\xi) \geq 0,\ \Im B(z,\xi) \leq n\Re\, A(z,\xi)[-n\Re\, A(z,\xi) + z].$$

If

$$A(z,\xi)\,[\frac{z^2 f''(z)}{f(z)} + \frac{z\, f'(z)}{f(z)} - \frac{z^2 (f'(z))^2}{(f(z))^2}] + B(z,\xi)\frac{z f'(z)}{f(z)} \prec\prec z,\ z \in U,\ ,\xi \in \bar{U}$$

then

$$\frac{z\, f'(z)}{f(z)} \prec \frac{1+z}{1-z}\ ,\ z \in U.$$

Acknowledgement: This work is partially supported by UKM-ST-06-FRGS0107-2009, MOHE Malaysia.

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