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# ON CLASSES OF FUNCTIONS IN THE UNIT DISK

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Abstract. New classes of analytic functions are defined by using differential operator of fractional power. We give some applications of the first order differential subordination and obtain sufficient conditions for normalized analytic functions.

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*Key words*: Differential operator; Fractional calculus; Subordination; operator, Subordination factor sequence.

## 1. Introduction and preliminaries

Recent years, many studies have been seen in the literature regarding the differential subordinations, both in real and complex plane. In this particular paper, we are using the concept of differential subordination in complex plane. We apply the first order linear differential subordination to obtain relations between new classes of analytic functions of fractional power.

In the theory of univalent functions, it concerned primarily with the class of functions $f$ analytic and univalent in the unit disk $U$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. This class of functions will be denoted by $\mathcal{S}$. Thus, each $f \in \mathcal{S}$ has a Taylor series expansion of the form

$$f(z) = z + a_2z^2 + a_3z^3 + ..., \ \ (|z| < 1).$$

Example of classes representing the functions of the class $\mathcal{S}$ are

$$f(z) = \frac{z}{1-z} = z + z^2 + z^3 + ...$$

and the Koebe function

$$f(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + ... \, .$$

In this article we consider the functions $F$ in the open disk $U := \{z \in \mathbb{C}, |z| < 1\}$, defined by

$$\begin{aligned} F(z) = \frac{z^\alpha}{(1-z)^\alpha} &= \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} z^{n+\alpha} \\ &= z^\alpha + \sum_{n=1}^{\infty} \frac{(\alpha)_n}{n!} z^{n+\alpha} \\ &= z^\alpha + \sum_{n=2}^{\infty} \frac{(\alpha)_{n-1}}{(n-1)!} z^{n+\alpha-1} \\ &= z + \sum_{n=2}^{\infty} \frac{(\alpha)_{n-1}}{(n-1)!} z^{n+\alpha-1}, \end{aligned}$$

where $\alpha \geq 1$ takes its values from the relation $\alpha := \frac{n+m}{m}, \ m \in \mathbb{N}$.

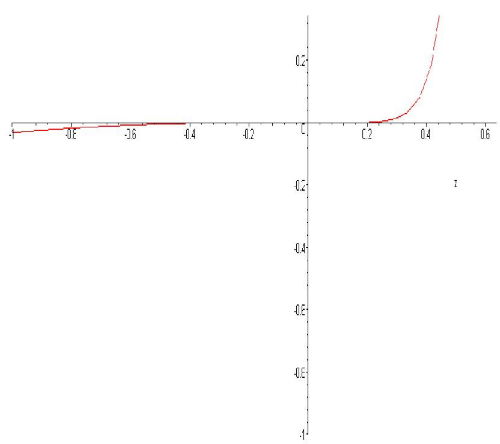

Figure 1: $F(z) \in \mathcal{A}_\alpha^+$

Let $\mathcal{A}_\alpha^+$ be the class of all normalized analytic functions $F$ take the form

$$F(z) = z + \sum_{n=2}^{\infty} a_{n,\alpha} z^{n+\alpha-1}, \ \ \alpha \geq 1,$$

where $a_{0,\alpha} = 0$, $a_{1,1} = 1$ satisfying $F(0) = 0$ and $F'(0) = 1$. And let $\mathcal{A}_\alpha^-$ be the class of all normalized analytic functions $F$) in the open disk $U$ take the form

$$F(z) = z - \sum_{n=2}^{\infty} a_{n,\alpha} z^{n+\alpha-1}, \ \ a_{n,\alpha} \geq 0; \ \ n = 2, 3, ...,$$

satisfying $F(0) = 0$ and $F'(0) = 1$.

In our present investigation, we require the following definitions:

**Definition 1.** *(Subordination Principle). For two functions $f$ and $g$ analytic in $U$, we say that the function $f$ is subordinate to $g$ in $U$ and write $f \prec g(z \in U)$, if there exists a Schwarz function $w$ analytic in $U$ with $w(0) = 0$, and $|w(z)| < 1$, such that $f(z) = g(w(z)), z \in U$. In particular, if the function $g$ is univalent in $U$, the above subordination is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$.*

**Definition 2.** *(Subordinating Factor Sequence). A sequence $\{b_{n,\alpha}\}_{n=1}^{\infty}, \alpha \geq 1$ of complex numbers is called a subordinating factor sequence if, whenever $F \in \mathcal{A}_\alpha^+$ is normalized analytic univalent and convex in $U$, we have the subordination given by*

$$\sum_{n=2}^{\infty} a_{n,\alpha} b_{n,\alpha} z^{n+\alpha-1} \prec F(z), \ \ (z \in U).$$

Now we define a differential operator as follows:

$$D_{\alpha,\lambda}^0 F(z) = F(z) = z + \sum_{n=2}^{\infty} a_{n,\alpha} z^{n+\alpha-1}, \ \ \alpha \geq 1, \ \ \lambda < \alpha$$

$$D_{\alpha,\lambda}^1 F(z) = (\lambda - \alpha + 1)F(z) + (\alpha - \lambda) z F'(z) = z + \sum_{n=2}^{\infty} [(\alpha - \lambda)(n + \alpha - 2) + 1] a_{n,\alpha} z^{n+\alpha-1}$$

$$\vdots$$

$$D_{\alpha,\lambda}^k F(z) = D(D^{k-1} F(z)) = z + \sum_{n=2}^{\infty} [(\alpha - \lambda)(n + \alpha - 2) + 1]^k a_{n,\alpha} z^{n+\alpha-1} \tag{1}$$

Let $\mathcal{A}$ be the class of analytic functions of the form $f(z) = z + a_2z^2 + \ldots$. Ali et al [1] have used the results of Bulboacâ [2] and obtain sufficient conditions for certain normalized analytic functions $f \in \mathcal{A}$ to satisfy

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z)$$

where $q_1$ and $q_2$ are given univalent functions in $U$ with $q_1(0) = q_2(0) = 1$. The main object of the present work is to apply a method based on the differential subordination in order to derive sufficient conditions for functions $F \in \mathcal{A}^+_\alpha$ and $F \in \mathcal{A}^-_\alpha$ to satisfy

$$\frac{z(D^k_{\alpha,\lambda}F(z))'}{D^k_{\alpha,\lambda}F(z)} \prec q(z) \tag{2}$$

where $q$ is a given univalent function in $U$ with $q(z) \neq 0$.

Next, we give applications for these results in fractional calculus. We shall need the following known results.

**Lemma 1.** *[3] Let $q$ be univalent in the unit disk $U$ and $\theta$ and $\phi$ be analytic in a domain $D$ containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) := zq'(z)\phi(q(z))$, $h(z) := \theta(q(z)) + Q(z)$. Suppose that*
*1. $Q(z)$ is starlike univalent in $U$, and*
*2. $\Re\frac{zh'(z)}{Q(z)} > 0$ for $z \in U$.*
*If $\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z))$ then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.*

**Lemma 2.***[4] Let $q(z)$ be convex univalent in the unit disk $U$ and $\psi$ and $\gamma \in \mathbb{C}$ with $\Re\{1+\frac{zq''(z)}{q'(z)}+\frac{\psi}{\gamma}\} > 0$. If $p(z)$ is analytic in $U$ and $\psi p(z)+\gamma zp'(z) \prec \psi q(z)+\gamma zq'(z)$, then $p(z) \prec q(z)$ and $q$ is the best dominant.*

**Lemma 3.***[5] The sequence $\{b_n\}_{n=1}^{\infty}$, is a subordinating factor sequence if and only if*

$$\Re\{1 + 2\sum_{n=1}^{\infty} b_n z^n\} > 0, \quad (z \in U). \tag{3}$$

## 2. Subordination results

In this section, we study the subordination between analytic functions.

**Theorem 1.** *Let the function $q$ be univalent in the unit disk $U$ such that $q(z) \neq 0$, and*

$$\Re\{1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}+\frac{b}{\gamma}q(z)+\frac{2c}{\gamma}q^2(z)+\frac{3d}{\gamma}q^3(z)\}>0,\ b,\ c,\in\mathbb{C},\ \gamma\neq 0. \quad (4)$$

*Suppose that $\frac{zq'(z)}{q(z)}$ is starlike univalent in $U$. If $F\in\mathcal{A}_\alpha^+$ satisfies the subordination*

$$a+b\frac{z(D^k_{\alpha,\lambda}F(z))'}{D^k_{\alpha,\lambda}F(z)}+c[\frac{z(D^k_{\alpha,\lambda}F(z))'}{D^k_{\alpha,\lambda}F(z)}]^2+d[\frac{z(D^k_{\alpha,\lambda}F(z))'}{D^k_{\alpha,\lambda}F(z)}]^3+\gamma[1+\frac{z(D^k_{\alpha,\lambda}F(z))''}{(D^k_{\alpha,\lambda}F(z))'}$$
$$-\frac{z(D^k_{\alpha,\lambda}F(z))'}{D^k_{\alpha,\lambda}F(z)}]\prec a+bq(z)+cq^2(z)+dq^3(z)+\gamma\frac{zq'(z)}{q(z)}.$$

*Then*

$$\frac{z(D^k_{\alpha,\lambda}F(z))'}{D^k_{\alpha,\lambda}F(z)}\prec q(z),\ \ z\in U,\ (D^k_{\alpha,\lambda}F(z)\neq 0)$$

*and $q$ is the best dominant.*

*Proof.* Let the function $p$ be defined by

$$p(z):=\frac{z(D^k_{\alpha,\lambda}F(z))'}{D^k_{\alpha,\lambda}F(z)},\ \ (D^k_{\alpha,\lambda}F(z)\neq 0,\ z\in U).$$

By setting

$$\theta(\omega):=a+b\omega+c\omega^2+d\omega^3\ \ \textit{and}\ \ \phi(\omega):=\frac{\gamma}{\omega},\ \ a\neq 0,$$

it can easily be observed that $\theta(\omega)$ is analytic in $\mathbb{C}$, $\phi(\omega)$ is analytic in $\mathbb{C}-\{0\}$ and that $\phi(\omega)\neq 0$, $\omega\in\mathbb{C}-\{0\}$. Also we obtain

$$Q(z)=zq'(z)\phi(q(z))=\gamma\frac{zq'(z)}{q(z)}\ \ \textit{and}$$

$$h(z)=\theta(q(z))+Q(z)=a+bq(z)+cq^2(z)+dq^3(z)+\gamma\frac{zq'(z)}{q(z)}.$$

It is clear that $Q(z)$ is starlike univalent in $U$,

$$\Re\{\frac{zh'(z)}{Q(z)}=\Re\{1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}+\frac{b}{\gamma}q(z)+\frac{2c}{\gamma}q^2(z)+\frac{3d}{\gamma}q^3(z)\}>0.$$

By a straightforward computation, we have

$$a+bp(z)+cp^2(z)+dp^3(z)+\gamma\frac{zp'(z)}{p(z)}=a+b\frac{z(D^k_{\alpha,\lambda}F(z))'}{D^k_{\alpha,\lambda}F(z)}+c[\frac{z(D^k_{\alpha,\lambda}F(z))'}{D^k_{\alpha,\lambda}F(z)}]^2$$
$$+d[\frac{z(D^k_{\alpha,\lambda}F(z))'}{D^k_{\alpha,\lambda}F(z)}]^3+\gamma[1+\frac{z(D^k_{\alpha,\lambda}F(z))''}{(D^k_{\alpha,\lambda}F(z))'}$$
$$-\frac{z(D^k_{\alpha,\lambda}F(z))'}{D^k_{\alpha,\lambda}F(z)}]\prec a+bq(z)+cq^2(z)+dq^3(z)+\gamma\frac{zq'(z)}{q(z)}.$$

Then by the assumption of the theorem we have that the assertion of the theorem follows by an application of Lemma 1.

**Corollary 1.** *Assume that (4) holds and $q$ is convex univalent in $U$. If $F\in\mathcal{A}^+_\alpha$ and*

$$a+b\frac{z(D^k_{\alpha,\lambda}F(z))'}{D^k_{\alpha,\lambda}F(z)}+c[\frac{z(D^k_{\alpha,\lambda}F(z))'}{D^k_{\alpha,\lambda}F(z)}]^2+d[\frac{z(D^k_{\alpha,\lambda}F(z))'}{D^k_{\alpha,\lambda}F(z)}]^3+\gamma[1+\frac{z(D^k_{\alpha,\lambda}F(z))''}{(D^k_{\alpha,\lambda}F(z))'}$$
$$-\frac{z(D^k_{\alpha,\lambda}F(z))'}{D^k_{\alpha,\lambda}F(z)}]\prec a+b\frac{1+Az}{1+Bz}+c[\frac{1+Az}{1+Bz}]^2+d[\frac{1+Az}{1+Bz}]^3+\gamma\frac{z(A-B)}{(1+Az)(1+Bz)}.$$

*Then*

$$\frac{z(D^k_{\alpha,\lambda}F(z))'}{D^k_{\alpha,\lambda}F(z)}\prec\frac{1+Az}{1+Bz},\ \ -1\le B<A\le 1$$

*and $q(z)=\frac{1+Az}{1+Bz}$ is the best dominant.*

**Corollary 2.** *Assume that (4) holds and $q$ is convex univalent in $U$. If $F\in\mathcal{A}^+_\alpha$ and*

$$a+b\frac{z(D^k_{\alpha,\lambda}F(z))'}{D^k_{\alpha,\lambda}F(z)}+c[\frac{z(D^k_{\alpha,\lambda}F(z))'}{D^k_{\alpha,\lambda}F(z)}]^2+d[\frac{z(D^k_{\alpha,\lambda}F(z))'}{D^k_{\alpha,\lambda}F(z)}]^3$$
$$+\gamma[1+\frac{z(D^k_{\alpha,\lambda}F(z))''}{(D^k_{\alpha,\lambda}F(z))'}-\frac{z(D^k_{\alpha,\lambda}F(z))'}{D^k_{\alpha,\lambda}F(z)}]$$
$$\prec a+b[\frac{1+z}{1-z}]^\mu+c[\frac{1+z}{1-z}]^{2\mu}+d[\frac{1+z}{1-z}]^{3\mu}+\frac{2\mu\gamma z}{(1+z)^2},$$

*for $z\in U$, $\mu\neq 0$, then*

$$\frac{z(D^k_{\alpha,\lambda}F(z))'}{D^k_{\alpha,\lambda}F(z)}\prec[\frac{1+z}{1-z}]^\mu$$

*and $q(z)=[\frac{1+z}{1-z}]^\mu$ is the best dominant.*

**Corollary 3.** *Assume that (4) holds and q is convex univalent in U. If* $F \in \mathcal{A}_\alpha^+$ *and*

$$a + b\frac{z(D^k_{\alpha,\lambda}F(z))'}{D^k_{\alpha,\lambda}F(z)} + c[\frac{z(D^k_{\alpha,\lambda}F(z))'}{D^k_{\alpha,\lambda}F(z)}]^2 + d[\frac{z(D^k_{\alpha,\lambda}F(z))'}{D^k_{\alpha,\lambda}F(z)}]^3$$
$$+ \gamma[1 + \frac{z(D^k_{\alpha,\lambda}F(z))''}{(D^k_{\alpha,\lambda}F(z))'} - \frac{z(D^k_{\alpha,\lambda}F(z))'}{D^k_{\alpha,\lambda}F(z)}]$$
$$\prec a + be^{\mu Az} + ce^{2\mu Az} + de^{3\mu Az} + \mu\gamma Az,$$

*for* $z \in U$, $\mu \neq 0$, *then*

$$\frac{z(D^k_{\alpha,\lambda}F(z))'}{D^k_{\alpha,\lambda}F(z)} \prec e^{\mu Az}$$

*and* $q(z) = e^{\mu Az}$ *is the best dominant.*
The next result can be found in [6].

**Corollary 4.** *Assume that* $k = 0$ *in Theorem 1, then*

$$\frac{z(F(z))'}{F(z)} \prec q(z), \ \ z \in U, \ F(z) \neq 0$$

*and* $q(z)$ *is the best dominant.*

**Theorem 2.** *Let the function q be convex univalent in the unit disk U such that*

$$\Re\{1 + \frac{zq''(z)}{q'(z)} + \frac{1}{\gamma}\} > 0, \ \ \gamma \neq 0. \tag{5}$$

*Suppose that* $\frac{z(D^k_{\alpha,\lambda}F(z))'}{D^k_{\alpha,\lambda}F(z)}$ *is analytic in U. If* $F \in \mathcal{A}_\alpha^-$ *satisfies the subordination*

$$\frac{z(D^k_{\alpha,\lambda}F(z))'}{D^k_{\alpha,\lambda}F(z)} + \gamma\frac{z(D^k_{\alpha,\lambda}F(z))'}{D^k_{\alpha,\lambda}F(z)}[1 + \frac{z(D^k_{\alpha,\lambda}F(z))''}{(D^k_{\alpha,\lambda}F(z))'} - \frac{z(D^k_{\alpha,\lambda}F(z))'}{D^k_{\alpha,\lambda}F(z)}] \prec q(z) + \gamma zq'(z).$$

*Then*

$$\frac{z(D^k_{\alpha,\lambda}F(z))'}{D^k_{\alpha,\lambda}F(z)} \prec q(z), \ \ (z \in U, \ D^k_{\alpha,\lambda}F(z) \neq 0)$$

*and q is the best dominant.*

*Proof.* Let the function $p$ be defined by

$$p(z) := \frac{z(D^k_{\alpha,\lambda}F(z))'}{D^k_{\alpha,\lambda}F(z)}, \ \ (D^k_{\alpha,\lambda}F(z) \neq 0, \ z \in U).$$

By setting $\psi = 1$, it can easily be observed that

$$p(z)+\gamma zp'(z) = \frac{z(D^k_{\alpha,\lambda}F(z))'}{D^k_{\alpha,\lambda}F(z)} + \gamma\frac{z(D^k_{\alpha,\lambda}F(z))'}{D^k_{\alpha,\lambda}F(z)}[1+\frac{z(D^k_{\alpha,\lambda}F(z))''}{(D^k_{\alpha,\lambda}F(z))'} - \frac{z(D^k_{\alpha,\lambda}F(z))'}{D^k_{\alpha,\lambda}F(z)}] \prec q(z)+\gamma zq'(z).$$

Then by the assumption of the theorem we have that the assertion of the theorem follows by an application of Lemma 2.

**Corollary 5.** *Assume that (5) holds and $q$ is convex univalent in $U$. If $F \in \mathcal{A}^-_\alpha$ and*

$$\frac{z(D^k_{\alpha,\lambda}F(z))'}{D^k_{\alpha,\lambda}F(z)} + \gamma\frac{z(D^k_{\alpha,\lambda}F(z))'}{D^k_{\alpha,\lambda}F(z)}[1+\frac{z(D^k_{\alpha,\lambda}F(z))''}{(D^k_{\alpha,\lambda}F(z))'} - \frac{z(D^k_{\alpha,\lambda}F(z))'}{D^k_{\alpha,\lambda}F(z)}] \prec \frac{1+Az}{1+Bz}+\frac{\gamma z(A-B)}{(1+Bz)^2}$$

*then*

$$\frac{z(D^k_{\alpha,\lambda}F(z))'}{D^k_{\alpha,\lambda}F(z)} \prec \frac{1+Az}{1+Bz}, \ \ -1 \le B < A \le 1$$

*and $q(z) = \frac{1+Az}{1+Bz}$ is the best dominant.*

**Corollary 6.** *Assume that (5) holds and $q$ is convex univalent in $U$. If $F \in \mathcal{A}^-_\alpha$ and*

$$\frac{z(D^k_{\alpha,\lambda}F(z))'}{D^k_{\alpha,\lambda}F(z)} + \gamma\frac{z(D^k_{\alpha,\lambda}F(z))'}{D^k_{\alpha,\lambda}F(z)}[1+\frac{z(D^k_{\alpha,\lambda}F(z))''}{(D^k_{\alpha,\lambda}F(z))'} - \frac{z(D^k_{\alpha,\lambda}F(z))'}{D^k_{\alpha,\lambda}F(z)}] \prec [\frac{1+z}{1-z}]^\mu+2\gamma\mu z\frac{(1+z)^{\mu-1}}{(1-z)^{\mu+1}}$$

*for $z \in U$, $\mu \neq 0$, then*

$$\frac{z(D^k_{\alpha,\lambda}F(z))'}{D^k_{\alpha,\lambda}F(z)} \prec [\frac{1+z}{1-z}]^\mu$$

*and $q(z) = [\frac{1+z}{1-z}]^\mu$ is the best dominant.*

**Corollary 7.** *Assume that (5) holds and $q$ is convex univalent in $U$. If $F \in \mathcal{A}^-_\alpha$ and*

$$\frac{z(D^k_{\alpha,\lambda}F(z))'}{D^k_{\alpha,\lambda}F(z)} + \gamma\frac{z(D^k_{\alpha,\lambda}F(z))'}{D^k_{\alpha,\lambda}F(z)}[1+\frac{z(D^k_{\alpha,\lambda}F(z))''}{(D^k_{\alpha,\lambda}F(z))'} - \frac{z(D^k_{\alpha,\lambda}F(z))'}{D^k_{\alpha,\lambda}F(z)}] \prec e^{\mu Az}+\mu\gamma Aze^{\mu Az}$$

*for $z \in U$, $\mu \neq 0$, then*

$$\frac{z(D^k_{\alpha,\lambda}F(z))'}{D^k_{\alpha,\lambda}F(z)} \prec e^{\mu Az}$$

*and $q(z) = e^{\mu Az}$ is the best dominant.*

The next result can be found in [6].

**Corollary 8.** *Assume that $k = 0$ in Theorem 2, then*

$$\frac{z(F(z))'}{F(z)} \prec q(z), \;\; z \in U, \, F(z) \neq 0$$

*and q) is the best dominant.*

**Theorem 3** *Let $F(z), \psi(z) \in \mathcal{A}_\alpha^+$ and $\psi(z)$ be univalent convex in U. Then*

$$(D^k_{\alpha,\lambda} F * \psi)(z) \prec \psi(z), \;\; (z \in U) \tag{6}$$

*and*

$$\Re\{F(z)\} > -\frac{1}{2[\alpha(\alpha-\lambda)+1]}, \;\; (z \in U). \tag{7}$$

*Proof.* Let

$$\psi(z) = z + \sum_{n=2}^{\infty} c_{n,\alpha} z^{n+\alpha-1}, \;\; 0 < \alpha \leq 1,$$

where $c_{0,\alpha} = 0$, $c_{1,1} = 1$ satisfying $\psi(0) = 0$ and $\psi'(0) = 1$. Then

$$(D^k_{\alpha,\lambda} F * \psi)(z) = z + \sum_{n=2}^{\infty} [(\alpha-\lambda)(n+\alpha-2)+1]^k a_{n,\alpha} c_{n,\alpha} z^{n+\alpha-1}.$$

By invoking Definition 2, the subordination (6) holds true if the sequence

$$b_{n,\alpha} := \{[(\alpha-\lambda)(n+\alpha-2)+1]^k a_{n,\alpha}\}_{n=1}^{\infty}$$

where $a_{0,\alpha} = 0$, $a_{1,1} = 1$ is a subordination factor sequence. By virtue of Lemma 3, this is equivalent to the inequality

$$\Re\{1 + \sum_{n=1}^{\infty} 2[(\alpha-\lambda)(n+\alpha-2)+1]^k a_{n,\alpha} z^{n+\alpha-1}\} > 0, \;\; (z \in U).$$

Since $[\alpha(\alpha-\lambda))+1]^k \leq [(\alpha-\lambda)(n+\alpha-2)+1]^k$, for all $n \geq 2$, we have the assertion (7).

## 3. Applications

In this section, we introduce some applications from Section 2 to fractional calculus. The fractional calculus is normed to the theory of integrals and derivatives in terms of integrals of arbitrary order (generalize the integer order).

Assume that $f(z) = \sum_{n=2}^{\infty} \varphi_n z^{n-1}$ and let us begin with the following definitions:

**Definition 3.**[7] *The fractional integral of order $\alpha$ is defined, for a function $f(z)$, by*

$$I_z^\alpha f(z) := \frac{1}{\Gamma(\alpha)} \int_0^z f(\zeta)(z-\zeta)^{\alpha-1} d\zeta; \quad \alpha \geq 1,$$

*where the function $f(z)$ is analytic in simply-connected region of the complex $z$-plane $(\mathbb{C})$ containing the origin and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $log(z-\zeta)$ to be real when$(z-\zeta) > 0$.* Note that (see [7,8])

$$I_z^\alpha z^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} z^{\mu+\alpha}, \ (\mu > -1).$$

Thus we have

$$I_z^\alpha f(z) = \sum_{n=2}^{\infty} a_n z^{n+\alpha-1}$$

where $a_n := \frac{\varphi_n \Gamma(n)}{\Gamma(n+\alpha)}$, for all $n = 2, 3, \ldots$. Implies $z + I_z^\alpha f(z) \in \mathcal{A}_\alpha^+$ and $z - I_z^\alpha f(z) \in \mathcal{A}_\alpha^-$ $(\varphi_n \geq 0)$, then we have the following results

**Theorem 4.** *For all $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ in Theorem 1, poses*

$$\frac{z D_{\alpha,\lambda}^k [1 + I_z^\alpha f'(z)]}{D_{\alpha,\lambda}^k [z + I_z^\alpha f(z)]} \prec q(z),$$

*and $q$ is the best dominant.*

*Proof.* Let the function $F(z)$ be defined by

$$F(z) := z + I_z^\alpha f(z), \quad z \in U, \, F(z) \neq 0.$$

Since $f(0) = 0$, one can verify that $[I_z^\alpha f(z)]' = I_z^\alpha f'(z)$ then we obtain the result [9].

**Theorem 5.***For all $k \in \mathbb{N}_0$ in Theorem 2, yields*

$$\frac{z D_{\alpha,\lambda}^k [1 - I_z^\alpha f'(z)]}{D_{\alpha,\lambda}^k [z - I_z^\alpha f(z)]} \prec q(z),$$

*and $q$ is the best dominant.*

*Proof.* Let the function $F(z)$ be defined by

$$F(z) := z - I_z^\alpha f(z), \quad z \in U, \, F(z) \neq 0.$$

Note that when $k = 0$ in Theorems 4 and 5 reduce to results obtained in [6].

**Remark 1.** For further reading about the classes of analytic functions of fractional power, see for examples [10,11].

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