Acta Universitatis Apulensis No. 24/2010
ISSN: 1582-5329 pp. 271-278

TENSOR PRODUCT OF ENDO-PERMUTATION MODULES

Ahmad M. Alghamdi And Makkiah S. Makki

Abstract. In this paper, we study some properties of the exterior tensor product on the category of modules. For this, we prove that the exterior tensor product of two permutation, endo-permutation, endo-trivial and endo-monomial modules are still permutation, endo-permutation endo- trivial and endo-monomial modules respectively. Also, we prove that the cap of an exterior tensor product of two modules equal the exterior tensor product of their caps. Also, we prove that the exterior tensor product of two Dade algebras is a Dade algebra.

2000 *Mathematics Subject Classification*: 20C20; 16S50.
Keywords: Tensor product, endo-permutation modules, Dade algebra.

1. Introduction

The concept of Endo-permutation modules for a p-group G for some prime number p is invented by E. Dade in two papers [8, 9]. It is a generalization of the permutation modules which is due to Green [10, 11]. E. Dade studied many properties of this class of modules and classified endo-permutation modules for abelian p-groups. The construction in [2] by J. Alperin is an important step for studying endo-permutation modules. However, the classification of endo-permutation modules over non- abelian p-groups took a long time to achieve. It has been completed by many authors and ended in 2004. See [15] for more details.

The importance of the class of endo-permutation modules can be seen when one studies the sources of simple G-modules in the case that G is a p-solvable group [14, Chapter 5] as well as when one studies the equivalence between blocks.

Another class of modules which has been introduced and studied by many authors is called endo-trivial modules [3, 9]. It is well known that each endo-trivial module is an endo-permutation module. This notion has been used as a tool to investigate and classify endo-permutation modules. The importance of the class of endo-trivial modules can be seen when one studies the equivalence of the stable category of a block [4]. Jon Carlson and Jacques Thevenaz classified this class of modules [5]. For a recent work on endo-trivial modules, the reader can consult the papers [6, 7].

Our purpose in this paper is to study the exterior tensor product of the category of these modules. Our motivation stems from the fact that the tensor product is an operation in mathematics which enables us to construct new objects from others.

For the coefficients, we consider for a prime number p a p-modular system (K, R, F). This means that R is a complete discrete valuation ring, K is the quotient field of R of characteristic zero and F is the field of residue of R which is an algebraically closed of characteristic p. For notation, we shall write $\mathcal{O}$ to denote R or F. We assume that all modules are free over $\mathcal{O}$ and finitely generated $\mathcal{O}G$-modules where G is an arbitrary finite group.

Our paper is organized as a sequence of sections each of which is concerned respectively to the tensor product of permutation modules, endo-permutation modules, endo-trivial modules, Dade's algebras and endo-monomial modules. We use the notation $\otimes$ to mean $\otimes_{\mathcal{O}}$.

2.Tensor product of permutation modules

Let G be a finite group and M be an $\mathcal{O}G$-module. We start with the following definition which is due to Green in [11].

Definition 2.1. *An $\mathcal{O}G$-module M is called an $\mathcal{O}G$-permutation module if it has an invariant $\mathcal{O}$-basis X under the action of G. In this case we write $M = \mathcal{O}X$.*

For any two finite groups G_1 and G_2, the exterior tensor product of two $\mathcal{O}G_1$ and $\mathcal{O}G_2$-modules can be defined as follows:

Definition 2.2. *If M_i is an $\mathcal{O}G_i$-module for $i = 1, 2$ the exterior tensor product of M_i is an $\mathcal{O}[G_1 \times G_2]$-module with underlying $\mathcal{O}$-module $M_1 \otimes M_2$ and the action of $G_1 \times G_2$ is given by: $(g_1, g_2).(m_1 \otimes m_2) = g_1m_1 \otimes g_2m_2$.*

Since all modules we are dealing with are finitely generated, any module M has a finite subset, say $X = \{x_i : i = 1, 2, ..., n\}$ such that any element $m \in M$ can be written uniquely as $m = \Sigma_{i=1}^{n} \alpha_i x_i$ with $\alpha_i \in \mathcal{O}$.

We call a finite set X a G-set if it is endowed with an action of G on it. The following lemma explains the exterior tensor product of two permutation modules. The proof is easy but, we include it for completeness.

Lemma 2.3. *If M_i is a permutation $\mathcal{O}G_i$-module, where $i = 1, 2$ then $M_1 \otimes M_2$ is a permutation $\mathcal{O}[G_1 \times G_2]$-module.*

Proof. Since each M_i is a permutation $\mathcal{O}G_i$-module for $i = 1, 2$ then from Definition 2.1, each M_i can be written in the the form $M_i = kX_i$ for $i = 1, 2$ where X_1 and X_2 are the invariant basis for M_1 and M_2 respectively. Now since $M_1 \otimes M_2$ is an $\mathcal{O}[G_1 \times G_2]$-module such that $M_1 \otimes M_2 = \mathcal{O}X_1 \otimes \mathcal{O}X_2$. Therefore,

$M_1 \otimes M_2 = \mathcal{O}[X_1 \times X_2]$. This means that the module $M_1 \otimes M_2$ has an invariant basis $X_1 \times X_2$. So, $M_1 \otimes M_2$ is a permutation $\mathcal{O}[G_1 \times G_2]$-module.

3. Tensor product of endo-permutation modules

Let G be a p-group for some prime number p. For any $\mathcal{O}G$-module M we denote by $End_{\mathcal{O}}(M)$ the endomorphism algebra of M. This algebra can be endowed with an $\mathcal{O}G$-module structure coming from the action of G by conjugation; that is if $g \in G$ and $\varphi \in End_{\mathcal{O}}(M)$, then ${}^{g}\varphi(m) = g.\varphi(g^{-1}.m)$ for all $m \in M$. Dade [8] defined the concept of an endo-permutation module as:

Definition 3.1. *An $\mathcal{O}G$-module M is called an endo-permutation module if $End_{\mathcal{O}}(M)$ is a permutation $\mathcal{O}G$-module.*

It is clear that each endo-permutation module is a permutation module.

Now we need the following two lemmas:

Lemma 3.2. *For any $\mathcal{O}G$-module M, we have $End_{\mathcal{O}}(M) \cong M \otimes M^*$ as an $\mathcal{O}G$-module, where $M^* = Hom(M, \mathcal{O})$ is the dual $\mathcal{O}G$-module and the tensor product is over $\mathcal{O}$.*

From the previous lemma, we see that M is an endo-permutation $\mathcal{O}G$-module if and only if $M \otimes M^*$ is a permutation $\mathcal{O}G$-module.

Lemma 3.3. *Let G_i be a finite group and M_i an $\mathcal{O}G_i$-module, where $i = 1, 2$. Then $End_{\mathcal{O}}(M_1 \otimes M_2) \cong End_{\mathcal{O}}(M_1) \otimes End_{\mathcal{O}}(M_2)$.*

We shall prove the main theorem in this section:

Theorem 3.4. *Let G_i be a finite group and M_i an endo-permutation $\mathcal{O}G_i$-module, where $i = 1, 2$. Then $M_1 \otimes M_2$ is an endo-permutation $\mathcal{O}[G_1 \times G_2]$-module.*

Proof. Since each M_i is an endo-permutation $\mathcal{O}G_i$-module for $i = 1, 2$ then from Definition 3.1 we have that $End_{\mathcal{O}}(M_i)$ is a permutation $\mathcal{O}G_i$-module. Now using Lemma 3.2 and Lemma 3.3 we see that the tensor module $M_1 \otimes M_2$ is an endo-permutation module as $End_{\mathcal{O}}(M_1 \otimes M_2)$ is a permutation module.

Recall that an indecomposable $\mathcal{O}G$-module M is called H-projective for some subgroup H of G if M is a direct summand of the induced $\mathcal{O}G$-module $Ind_H^G(N)$ for some $\mathcal{O}H$-module N. A minimal subgroup of the collection of all subgroups of G for which M is H-projective is called a vertex of M. It turns out that the vertex is a p-subgroup of G. We write Vertex(M) for the vertex group of M.

We are interested in indecomposable endo-permutation $\mathcal{O}G$-modules with maximal vertex G, because they are the ones that appear in representation theory. All endo-permutation modules can be described from the knowledge of the indecomposable ones having maximal vertex (see [14] for more details).

Definition 3.5. *An endo-permutation $\mathcal{O}G$-module M is said to be capped if it has at least one indecomposable direct summand with vertex G. Such indecomposable direct summand is called the cap of M and is denoted by M^c.*

In particular, if M is indecomposable, then M is capped if and only if it has vertex G.

We shall use the following result which is due to B. Külshammer [13] for dealing with the tensor product of capped module.

Lemma 3.6. *Suppose that M_i is an indecomposable $\mathcal{O}G_i$-module; i=1, 2 with vertex $(M_i) = V_i$. Then $M_1 \otimes M_2$ is an indecomposable $\mathcal{O}[G_1 \times G_2]$-module with vertex $(M_1 \otimes M_2) = V_1 \times V_2$.*

In the following theorem we shall prove a result about the relationship between the tensor product of the cap of two endo-permutation modules and the cap of their tensor product.

Theorem 3.7. *Let M_i be an endo-permutation $\mathcal{O}G_i$-module for $i = 1, 2$. Suppose that M_i^c is the cap of M_i. Then $M_1^c \otimes M_2^c = (M_1 \otimes M_2)^c$ where $(M_1 \otimes M_2)^c$ is the cap of the endo-permutation $\mathcal{O}[G_1 \times G_2]$-module $M_1 \otimes M_2$.*

Proof. Since M_i^c is the cap of M_i that is an indecomposable direct summand of M_i with vertex G_i, for $i = 1, 2$ Lemma 3.6 yields that $M_1^c \otimes M_2^c$ is an indecomposable $\mathcal{O}[G_1 \times G_2]$-module with vertex $G_1 \times G_2$. However, $M_1 \otimes M_2$ has a unique indecomposable direct summand with vertex $G_1 \times G_2$. So, the result is complete.

4.The tensor product of endo-trivial modules

We continue to assume that G is a p-group for some prime number p and $\mathcal{O}$ is either an algebraically closed field of characteristic p or a complete discrete valuation ring of characteristic zero.

Definition 4.1. *An $\mathcal{O}G$-module M is called endo-trivial if there exists a projective $\mathcal{O}G$-module F such that $End_{\mathcal{O}}(M) \cong \mathcal{O} \oplus F$ as an $\mathcal{O}G$-module.*

Theorem 4.2. *If M_i is an endo-trivial $\mathcal{O}G_i$-modules, where $i = 1, 2$ then $M_1 \otimes M_2$ is an endo-trivial $\mathcal{O}[G_1 \times G_2]$-module.*

Proof. Since M_i is an endo-trivial $\mathcal{O}G_i$-module, there is a projective $\mathcal{O}G_i$-module F_i such that $End_{\mathcal{O}}(M_i) \cong \mathcal{O} \bigoplus F_i$ for $i = 1, 2$. By Lemma 3.3, $End_{\mathcal{O}}(M_1 \otimes M_2) \cong End_{\mathcal{O}}(M_1) \otimes End_{\mathcal{O}}(M_2)$. So,

$$End_{\mathcal{O}}(M_1 \otimes M_2) \cong (\mathcal{O} \oplus F_1) \otimes (\mathcal{O} \oplus F_2).$$

Therefore,

$$End_{\mathcal{O}}(M_1 \otimes M_2) \cong (\mathcal{O} \otimes \mathcal{O}) \oplus (F_1 \otimes \mathcal{O}) \oplus (\mathcal{O} \otimes F_2) \oplus (F_1 \otimes F_2).$$

However, as $\mathcal{O}[G_1 \times G_2]$-isomorphism, we have $\mathcal{O} \otimes \mathcal{O} \cong \mathcal{O}$, $F_1 \otimes \mathcal{O} \cong F_1$ and $\mathcal{O} \otimes F_2 \cong F_2$. Also $F' := F_1 \otimes F_2$ is a projective $\mathcal{O}[G_1 \times G_2]$-module. Hence, $End_{\mathcal{O}}(M_1 \otimes M_2) \cong \mathcal{O} \oplus F_1 \oplus F_2 \oplus F'$. It follows that $End_{\mathcal{O}}(M_1 \otimes M_2) \cong \mathcal{O} \oplus F$, where $F := F_1 \oplus F_2 \oplus F'$ is a projective $\mathcal{O}[G_1 \times G_2]$-module. By the definition, this means that $M_1 \otimes M_2$ is an endo-trivial $\mathcal{O}[G_1 \times G_2]$-module.

5.The tensor product of Dade algebras

A G-algebra over $\mathcal{O}$ is an $\mathcal{O}$-algebra endowed with an action of the group G by algebra automorphisms such that $\Psi(g)(a) = {}^g a$ for $a \in A, \Psi \in aut(A)$. For any G-algebra A we define the set of G-fixed elements of A by $A^G = \{a \in A :$
${}^g a = a \ \ \forall \ \ g \in G\}$.

Let H be a subgroup of G. We consider the relative trace map $Tr_H^G : A^H \to A^G$ such that $Tr_H^G(a) = \sum_{t\in T} {}^t a$, where T is a set of representatives of the left cosets of H in G. It is clear that the image of Tr_H^G is an ideal of A^G. For technical reason and as we use p-modular system, we shall write the sum of the image of the trace map and the ideal $\wp A^H$ as A_H^G, where $\wp$ is the unique maximal ideal in $\mathcal{O}$. We define the Brauer quotient as $A(H) := A^H / \sum_{K<H} A_K^H$.

Definition 5.1. *Dade G-algebra A is an $\mathcal{O}$-simple permutation G-algebra such that $A(G) \neq 0$.*

Suppose that G_1 and G_2 are two finite groups and let A_i; $i = 1, 2$ be a G_i-algebra over $\mathcal{O}$, the tensor algebra $A_1 \bigotimes A_2$ can be regarded as a $G_1 \times G_2$-algebra by the action ${}^{(g_1,g_2)}(a_1 \otimes a_2) = {}^{g_1}a_1 \otimes {}^{g_2}a_2$, for all $(g_1, g_2) \in G_1 \times G_2, a_i \in A_i$.

Now to introduce the main result in this section we recall the following lemma about the $H_1 \times H_2$-fixed elements of $A_1 \bigotimes A_2$ as well as the image of the tensor of the relative trace maps. For more details, see [1, Lemma 2.1 and Lemma 2.3].

Lemma 5.2. *Assume that $K_i \leq H_i \leq G_i$ for $i = 1, 2$. Then*

$$(A_1 \otimes A_2)^{H_1 \times H_2} \cong A_1^{H_1} \otimes A_2^{H_2},$$

and

$$(A_1 \otimes A_2)_{K_1 \times K_2}^{H_1 \times H_2} \cong A_1{}_{K_1}^{H_1} \otimes A_2{}_{K_2}^{H_2}.$$

Theorem 5.3. *The tensor product of any two Dade algebras is a Dade algebra.*

Proof. Suppose that A_i is a Dade G_i-algebra for $i = 1, 2$. This means that each A_i is an $\mathcal{O}$-simple permutation G_i-algebra such that $A_i(G_i) \neq 0$. It is clear that the tensor product $A_1 \otimes A_2$ is an $\mathcal{O}$-simple permutation $G_1 \times G_2$-algebra.

Now let us assume that $A_1 \otimes A_2(G_1 \times G_2) = 0$. Then we have

$$(A_1 \otimes A_2)^{G_1 \times G_2} = \sum_{H_1 \times H_2 \leq G_1 \times G_2} [A_1 \otimes A_2]^{G_1 \times G_2}_{H_1 \times H_2}.$$

Using Lemma 5.2, we see that

$$A_1^{G_1} \otimes A_2^{G_2} = (\sum_{H_1 \leq G_1} A_1{}^{G_1}_{H_1}) \otimes (\sum_{H_2 \leq G_2} A_2{}^{G_2}_{H_2}).$$

It follows that, $A_i^{G_i} = \sum_{H_i \leq G_i} A_i{}^{G_i}_{H_i}$. This means that $A_i(G_i) = 0$, which is a contradiction. Hence, the result is the tensor product of any two Dade algebras is a Dade algebra.

6.The tensor product of the monomial modules

The notion of monomial representations arises for induction from linear representations. Let G be a finite group and H a subgroup of G. Consider the linear characters of H which are the homomorphisms from H to the multiplicative group of $\mathcal{O}$. Let N be an $\mathcal{O}H$-module. Assume that H acts on N via the linear $\mathcal{O}$-characters of H. We say that N is an $\mathcal{O}H$-module of $\mathcal{O}$-rank one if N is isomorphic to $\mathcal{O}$ as $\mathcal{O}$-module and H acts as follows: for $h \in H$ and for $n \in N$ we have $h{\cdot}\, n = \lambda_h n$ for some $\lambda_h \in \mathcal{O}$. Note that the action is well defined as we have a homomorphism from H to the group of units of $\mathcal{O}$. For more details see [12].

Definition 6.1. *We say that M is a monomial $\mathcal{O}G$-module if $M = Ind_H^G(N)$ for some subgroup H of G and some $\mathcal{O}$-rank one $\mathcal{O}H$-module N.*

Now consider H_i to be a subgroup of a finite group G_i, for $i = 1, 2$. We would like to study the tensor product of two $\mathcal{O}$-rank one modules.

Lemma 6.2. *Let N_i be an $\mathcal{O}$-rank one $\mathcal{O}H_i$-module for $i = 1, 2$. The tensor product $N_1 \otimes N_2$ is an $\mathcal{O}$-rank one $\mathcal{O}[H_1 \times H_2]$-module.*

Proof. It is clear that as an $\mathcal{O}$-module, $N_1 \otimes N_2 \cong \mathcal{O} \otimes \mathcal{O} \cong \mathcal{O}$. For the action, if $h_i \in H_i$ and $\lambda_i \in \mathcal{O}$, for $i = 1, 2$, with $h_i{\cdot}\, n_i = \lambda_i n_i$, we see that the element (h_1, h_2) in $H_1 \times H_2$ acts on $(n_1 \otimes n_2) \in N_1 \otimes N_2$ in such way

$$(h_1, h_2){\cdot}\, (n_1 \otimes n_2) = \lambda_1 n_1 \otimes \lambda_2 n_2 = \lambda_1 \lambda_2 (n_1 \otimes n_2) \in N_1 \otimes N_2.$$

We conclude that $N_1 \otimes N_2$ is an $\mathcal{O}$-rank one $\mathcal{O}[H_1 \times H_2]$-module.

The following proposition relates two monomial modules under the exterior tensor product.

Proposition 6.3. *Let M_i be a monomial $\mathcal{O}G_i$-module, for $i = 1, 2$. Then we have $M_1 \otimes M_2$ is a monomial $\mathcal{O}[G_1 \times G_2]$-module.*

Proof. The assumption that M_i is a monomial $\mathcal{O}G_i$-module for $i = 1, 2$ means that there is a subgroup H_i of G_i such that $M_i = Ind_{H_i}^{G_i}(N_i)$, for some $\mathcal{O}$-rank one $\mathcal{O}H_i$-module, for $i = 1, 2$. Using an analogue theory for Lemma 5.2 for modules, we see that

$$M_1 \otimes M_2 = Ind_{H_1}^{G_1}(N_1) \otimes Ind_{H_2}^{G_2}(N_2) = Ind_{H_1 \times H_2}^{G_1 \times G_2}(N_1 \otimes N_2).$$

Now Lemma 6.2 completes the proof and $M_1 \otimes M_2$ is a monomial $\mathcal{O}[G_1 \times G_2]$-module.

A similar theory for endo-permutation modules can be done for monomial modules. Let us introduce the following definition.

Definition 6.4. *Let M be an $\mathcal{O}G$-module. We say that M is an endo-monomial $\mathcal{O}G$-module if $End_{\mathcal{O}}(M)$ is a monomial $\mathcal{O}G$-module.*

Since in this paper we concern to generalize some results in the category of modules to the exterior tensor product, we have the following theorem.

Theorem 6.5. *Let M_i be an endo-monomial $\mathcal{O}G_i$-module, for $i = 1, 2$. Then we have $M_1 \otimes M_2$ is an endo-monomial $\mathcal{O}[G_1 \times G_2]$-module.*

Proof. By Definition 6.4, $End_{\mathcal{O}}(M_i)$ is a monomial $\mathcal{O}G_i$-module, for $i = 1, 2$. By Proposition 6.3, $End_{\mathcal{O}}(M_1) \otimes End_{\mathcal{O}}(M_2)$ is a monomial $\mathcal{O}[G_1 \times G_2]$-module, for $i = 1, 2$. Now Lemma 3.3 implies that $End_{\mathcal{O}}(M_1 \otimes M_2)$ is a a monomial $\mathcal{O}[G_1 \times G_2]$-module. So, $M_1 \otimes M_2$ is an endo-monomial $\mathcal{O}[G_1 \times G_2]$-module.

References

[1] Ahmad. M. Alghamdi and Ahmed. A. Khammash, *Defect groups of tensor modules*, Journal of Pure and Applied Algebra, 167, (2002), 165 - 173.

[2] J. Alperin, *A construction of endo-permutation modules*, Journal of Group Theory, 4, (2001), 3 - 10.

[3] J. Alperin, *Lifting endo-trivial modules*, Journal of Group Theory, 4, (2001), 1-2.

[4] J. Carlson and J. Thévenaz, *Torsion Endo-trivial modules*, Algebras and Representation, 3, (2000), 303 - 335.

[5] J. Carlson and J. Thévenaz, *The classification of endo-trivial modules*, Inventiones mathematica, 158, (2004), 389 - 411.

[6] J. Carlson and N. Mazza and D. Nakano, *Endo-trivial modules for the symmetric and alternating groups*, Proceeding of Edinburgh Mathematical Society, 52, (2009), 45 - 66.

[7] J. Carlson and N. Mazza and D. Nakano, *Endo-trivial modules for finite groups of lie type*, J. Reine Angew. Math., 595, (2006), 93 - 120.

[8] E. C. Dade, *Endo-permutation modules over p-groups I*, Ann.Math., 107, (1978), 459-494.

[9] E. C. Dade, *Endo-permutation modules over p-groups II*, Ann.Math., 108, (1978), 317-346.

[10] J. Green, *On the indecomposable representations of a finite group*, Math. Z., 70, (1959), 430 - 445.

[11] J. Green, *Blocks of modular representations*, Math. Z., 79, (1962), 100 - 115.

[12]R. Hartmann, *Endo-monomial modules over p-groups and their classification in the abelian case*, Journal of Algebra, 274, (2004), (2), 564-586.

[13] B. Külshammer, *Some indecomposable modules and their vertices*, Journal of Pure and Applied Algebra, 86, (1993), 65 - 73.

[14] J. Thévenaz, *G-Algebras and Modular Representation Theory*, Oxford Science Publications, Oxford, 1995.

[15] J. Thévenaz, *Endo-permutation modules, a guided tour*, http://infoscience.epfl.ch /record/130470/files/, (2006), 1 - 31.

Ahmad M. Alghamdi
Department of Mathematics,
Faculty of Sciences
Umm Alqura University,
P.O. Box 14035, Makkah 21955,Saudi Arabia.
email:*amghamdi@uqu.edu.sa*

Makkiah S. Makki
Department of Mathematics
Faculty of Applied Sciences for Girls,
Umm Alqura University
P.O. Box 20356 , Makkah 21955, Saudi Arabia.
email:*msmakki@uqu.edu.sa*