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∂-HAUSDORFF BICLOSURE SPACES

Chawalit Boonpok

Abstract. The purpose of the present paper is to introduce the concept of ∂-Hausdorff biclosure spaces and investigate some of their characterizations.

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1. Introduction

Generalized closed sets, briefly g-closed sets, in a topological space were introduced by N. Levine [7] in order to extend some important properties of closed sets to a larger family of sets. For instance, it was shown that compactness, normality and completeness in a uniform apace are inherited by g-closed subsets. K. Balachandran, P. Sundaram and H. Maki [1] introduced the notion of generalized continuous maps, briefly g-continuous maps, by using g-closed sets and studied some of their properties.

Closure spaces were introduced by E.Čech in [4] and then studied by many authors, see e.g. [5], [8] and [9]. C. Boonpok and J. Khampakdee [2] introduced a new class of closed sets in closure spaces, as for generality, between the class of closed sets and the class of generalized closed sets. Using the concept of ∂-closed sets, introduced two new kinds of spaces, namely $T'_{\frac{1}{2}}$-spaces and $T''_{\frac{1}{2}}$-spaces. The two kinds of spaces are investigated.

J. C. Kelly [6] introduced the notion of bitopological spaces. Such spaces are equipped with two arbitrary topologies. Furthermore, Kelly extended some of the standard results of separation axioms in a topological space to a bitopological space. Thereafter, a large number of papers have been written to generalize topological concepts to bitopological setting. In this, paper we introduce and study the concept of ∂-Hausdorff biclosure spaces.

2. Preliminaries

An operator $u : P(X) \rightarrow P(X)$ defined on the power set $P(X)$ of a set X satisfying the axioms :

(N1) $u\emptyset = \emptyset$,

(N2) $A \subseteq uA$ for every $A \subseteq X$,

(N3) $A \subseteq B \Rightarrow uA \subseteq uB$ for all $A, B \subseteq X$.

is called a *closure operator* and the pair (X, u) is called a *closure space*. For short, the space will be noted by X as well, and called a *closure space*. A closure operator u on a set X is called *idempotent* if $uA = uuA$ for all $A \subseteq X$. A subset A is *closed* in the closure space (X, u) if $uA = A$ and it is *open* if its complement is closed. The empty set and the whole space are both open and closed.

Let (X, u_1) and (X, u_2) be closure spaces. The closure u_1 is said to be *finer* than the closure u_2, or u_2 is said to be *coarser* than u_1, by symbols $u_1 \leq u_2$, if $u_2A \supseteq u_1A$ for every $A \subseteq X$. The relation $\leq$ is a partial order on the set of all closure operators on X.

A closure space (Y, v) is said to be a *subspace* of (X, u) if $Y \subseteq X$ and $vA = uA \cap Y$ for each subset $A \subseteq Y$. If Y is closed in (X, u), then the subspace (Y, v) of (X, u) is said to be closed too. A closure space (X, u) is said to be a T_0*-space* if, for any pair of points $x, y \in X$, from $x \in u\{y\}$ and $y \in u\{x\}$ it follows that $x = y$, and it is called a $T_{\frac{1}{2}}$*-space* if each singleton subset of X is closed or open. Let (Y, v) be a closed subspace of (X, u). If F is a closed subset of (Y, v), then F is a closed subset of (X, u). Let (X, u) and (Y, v) be closure spaces. A map $f : (X, u) \rightarrow (Y, v)$ is said to be *continuous* if $f(uA) \subseteq vf(A)$ for every subset $A \subseteq X$.

One can see that a map $f : (X, u) \rightarrow (Y, v)$ is continuous if and only if $uf^{-1}(B) \subseteq f^{-1}(vB)$ for every subset $B \subseteq Y$. Clearly, if $f : (X, u) \rightarrow (Y, v)$ is continuous, then $f^{-1}(F)$ is a closed subset of (X, u) for every closed subset F of (Y, v). Let (X, u) and (Y, v) be closure spaces. A map $f : (X, u) \rightarrow (Y, v)$ is said to be *closed* (resp. *open*) if $f(F)$ is a closed (resp. open) subset of (Y, v) whenever F is a closed (resp. open) subset of (X, u).

The *product* of a family $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ of closure spaces, denoted by $\prod_{\alpha \in I}(X_\alpha, u_\alpha)$, is the closure space $\big(\prod_{\alpha \in I} X_\alpha, u\big)$ where $\prod_{\alpha \in I} X_\alpha$ denotes the cartesian product of sets X_α, $\alpha \in I$, and u is the closure operator generated by the projections $\pi_\alpha : \prod_{\alpha \in I}(X_\alpha, u_\alpha) \rightarrow (X_\alpha, u_\alpha)$, $\alpha \in I$, i.e., is defined by $uA = \prod_{\alpha \in I} u_\alpha \pi_\alpha(A)$ for each $A \subseteq \prod_{\alpha \in I} X_\alpha$.

Clearly, if $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ is a family of closure spaces, then the projection map $\pi_\beta : \prod_{\alpha\in I}(X_\alpha, u_\alpha) \to (X_\beta, u_\beta)$ is closed and continuous for every $\beta \in I$.

Proposition 2.1. *Let $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ be a family of closure spaces and let $\beta \in I$. Then F is a closed subset of (X_β, u_β) if and only if $F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$.*

Proof. Let F be a closed subset of (X_β, u_β). Since π_β is continuous, $\pi_\beta^{-1}(F)$ is a closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$. But $\pi_\beta^{-1}(F) = F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$, hence $F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$.

Conversely, let $F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ be a closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$. Since π_β is closed, $\pi_\beta\Big(F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha\Big) = F$ is a closed subset of (X_β, u_β).

The following statement is evident :

Proposition 2.2. *Let $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ be a family of closure spaces and let $\beta \in I$. Then G is an open subset of (X_β, u_β) if and only if $G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is an open subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$.*

Definition 2.3.*[2] Let (X, u) be a closure space. A subset $A \subseteq X$ is called a* generalized closed *set, briefly a* g-closed *set, if $uA \subseteq G$ whenever G is an open subset of (X, u) with $A \subseteq G$. A subset $A \subseteq X$ is called a* generalized open *set, briefly a* g-open *set, if its complement is g-closed.*

Definition 2.4. *[2] Let (X, u) be a closure space. A subset $A \subseteq X$ is called a* ∂-closed *set, if $uA \subseteq G$ whenever G is a g-open subset of (X, u) with $A \subseteq G$. A subset $A \subseteq X$ is called a* ∂-open *set if its complement is ∂-closed.*

Remark 2.5. For a subset A of a closure space (X, u), the following implications hold :

$$A \text{ is closed} \Rightarrow A \text{ is } \partial\text{-closed} \Rightarrow A \text{ is g-closed}$$

None of these implications is reversible as shown by the following examples.

Example 2.6. Let $X = \{1,2,3,4\}$ and define a closure operator u on X by $u\emptyset = \emptyset$, $u\{1\} = \{1,3\}$, $u\{2\} = \{2,3\}$, $u\{3\} = u\{4\} = u\{3,4\} = \{3,4\}$ and $u\{1,2\} = u\{1,3\} = u\{1,4\} = u\{2,3\} = u\{2,4\} = u\{1,2,3\} = u\{1,2,4\} = u\{2,3,4\} = u\{1,3,4\} = uX = X$. Then $\{1,2,3\}$ is ∂-closed set but it is not closed.

Example 2.7. Let $X = \{1,2\}$ and define a closure operator u on X by $u\emptyset = \emptyset$ and $u\{1\} = u\{2\} = uX = X$. Then $\{1\}$ is g-closed but it is not ∂-closed.

Proposition 2.8. *Let (X,u) be a closure space and let $A, B \subseteq X$. If A is closed and B is g-closed, then $A \cap B$ is g-closed.*

Proof. Let G be an open subset of (X,u) such that $A \cap B \subseteq G$. Then $B \subseteq (X - A) \cup G$. Since $(X - A) \cup G$ is open, $uB \subseteq (X - A) \cup G$. Consequently, $A \cap uB \subseteq G$. Since A is closed, $u(A \cap B) \subseteq uA \cap uB = A \cap uB \subseteq G$. Hence, $A \cap B$ is g-closed.

Proposition 2.9. [2] *Let $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ be a family of closure spaces and let $\beta \in I$. Then F is a ∂-closed subset of (X_β, u_β) if and only if $F \times \prod\limits_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a ∂-closed subset of $\prod\limits_{\alpha \in I} (X_\alpha, u_\alpha)$.*

Proposition 2.10. [2] *Let $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ be a family of closure spaces and let $\beta \in I$. Then G is a ∂-open subset of (X_β, u_β) if and only if $G \times \prod\limits_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a ∂-open subset of $\prod\limits_{\alpha \in I} (X_\alpha, u_\alpha)$.*

Proposition 2.11. [2] *Let $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ be a family of closure spaces. For each $\beta \in I$, let $\pi_\beta : \prod\limits_{\alpha \in I} X_\alpha \to X_\beta$ be the projection map. Then*

(i) If F is a ∂-closed subset of $\prod\limits_{\alpha \in I} (X_\alpha, u_\alpha)$, then $\pi_\beta(F)$ is a ∂-closed subset of (X_β, u_β).

(ii) If F is a ∂-closed subset of (X_β, u_β), then $\pi_\beta^{-1}(F)$ is a ∂-closed subset of $\prod\limits_{\alpha \in I} (X_\alpha, u_\alpha)$.

Definition 2.12. *Let (X,u) and (Y,v) be closure spaces. A map $f : (X,u) \to (Y,v)$ is called* ∂-irresolute, *if $f^{-1}(F)$ is a ∂-closed subset of (X,u) for every ∂-closed subset F of (Y,v).*

Clearly, a map $f : (X, u) \to (Y, v)$ is ∂-irresolute if and only if $f^{-1}(G)$ is a ∂-open subset of (X, u) for every ∂-open subset G of (Y, v).

Definition 2.13. *[3] A biclosure space is a triple (X, u_1, u_2) where X is a set and u_1, u_2 are two closure operators on X.*

Definition 2.14. *[3] A subset A of a biclosure space (X, u_1, u_2) is called closed if $u_1u_2A = A$. The complement of closed set is called open.*

Clearly, A is a closed subset of a biclosure space (X, u_1, u_2) if and only if A is both a closed subset of (X, u_1) and (X, u_2).

Let A be a closed subset of a biclosure space (X, u_1, u_2) . The following conditions are equivalent:

(i) $u_2u_1A = A$,

(ii) $u_1A = A$, $u_2A = A$.

Definition 2.15. *[3] Let (X, u_1, u_2) be a biclosure space. A biclosure space (Y, v_1, v_2) is called a subspace of (X, u_1, u_2) if $Y \subseteq X$ and $v_iA = u_iA \cap Y$ for each $i \in \{1, 2\}$ and each subset $A \subseteq Y$.*

Definition 2.16. *Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces and let $i \in \{1, 2\}$. A map $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ is called i-∂-irresolute if the map $f : (X, u_i) \to (Y, v_i)$ is ∂-irresolute. A map f is called ∂-irresolute if f is i-∂-irresolute for each $i \in \{1, 2\}$.*

3. ∂-Hausdorff biclosure spaces

In this section, we introduce the concept of ∂-Hausdorff biclosuure spaces and study some of their properties.

Definition 3.1. *A biclosure space (X, u_1, u_2) is said to be ∂-Hausdorff biclosure space if, whenever x and y are distinct points of X there exists ∂-open subset U of (X, u_1) and ∂-open subset V of (X, u_2) such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.*

Example 3.2. Let $X = \{a, b\}$ and define a closure operator u_1 on X by $u_1\emptyset = \emptyset$, $u_1\{a\} = u_1\{b\} = u_1X = X$. Define a closure operator u_2 on X by $u_2\emptyset = \emptyset$, $u_2\{a\} = u_2\{b\} = u_2X = X$. Then (X, u_1, u_2) is a ∂-Hausdorff biclosure space.

Proposition 3.3. *Let (X, u_1, u_2) be a biclosure space and let $A, B \subseteq X$. If A is a closed subset of (X, u_1, u_2) and B is both a ∂-closed subset of (X, u_1) and (X, u_2), then $A \cap B$ is both a ∂-closed subset of (Y, v_1) and (Y, v_2).*

Proof. Let G be a g-open subset of (X, u_1) such that $A \cap B \subseteq G$. Then $B \subseteq (X - A) \cup G$. Since $(X - A) \cup G$ is a g-open subset of (X, u_1), $u_1 B \subseteq (X - A) \cap G$. Consequently, $A \cap u_1 B \subseteq G$. Since A is a closed subset of (X, u_1), $u_1(A \cap B) \subseteq u_1 A \cap u_1 B = A \cap u_1 B \subseteq G$. Hence, $A \cap B$ is a ∂-closed subset of (X, u_1). Similarly, if A is a closed subset of (X, u_1, u_2) and B is a ∂-closed subset of (X, u_2), then $A \cap B$ is a ∂-closed subset of (Y, v_2).

Lemma 3.4. *Let (X, u_1, u_2) be a biclosure space and let (Y, v_1, v_2) be a closed subspace of (X, u_1, u_2). If G is both a ∂-open subset of (X, u_1) and (X, u_2), then $G \cap Y$ is both a ∂-open subset of (Y, v_1) and (Y, v_2).*

Proof. Let G be a ∂-open subset of (X, u_1). Then $X - G$ is a ∂-closed subset of (X, u_1). Since Y is a closed subset of (X, u_1), $(X - G) \cap Y$ is a ∂-closed subset of (X, u_1). But $(X - G) \cap Y = Y - (G \cap Y)$. Therefore, $Y - G \cap Y$ is a ∂-closed subset of (X, u_1). Hence, $G \cap Y$ is a ∂-open subset of (X, u_1). Similarly, if G is a ∂-open subset of (X, u_2), then $G \cap Y$ is a ∂-open subset of (Y, v_2).

Proposition 3.5. *Let (X, u_1, u_2) be a biclosure space and let (Y, v_1, v_2) be a closed subspace of (X, u_1, u_2). If (X, u_1, u_2) is a ∂-Hausdorff biclosure space, then (Y, v_1, v_2) is a ∂-Hausdorff biclosure space.*

Proof. Let y and y' be any two distinct points of Y. Then y and y' are distinct points of X. Since (X, u_1, u_2) is a ∂-Haurdorff biclosure space, there exists a disjoint ∂-open subset U of (X, u_1) and ∂-open subset V of (X, u_2) containing y and y', respectively. Consequently, $y \in U \cap Y$, $y' \in V \cap Y$ and $(U \cap Y) \cap (V \cap Y) = \emptyset$. By Lemma 3.4, $U \cap Y$ is a ∂-open subset of (Y, v_1) and $V \cap Y$ is ∂-open subset of (Y, v_2). Hence, (Y, v_1, v_2) is a ∂-Hausdorff biclosure space.

Proposition 3.6. *Let $\{(X_\alpha, u_\alpha^1, u_\alpha^2) : \alpha \in I\}$ be a family of biclosure spaces. Then $\prod_{\alpha \in I}(X_\alpha, u_\alpha^1, u_\alpha^2)$ is a ∂-Hausdorff biclosure space if and only if $(X_\alpha, u_\alpha^1, u_\alpha^2)$ is a ∂-Hausdorff biclosure space for each $\alpha \in I$.*

Proof. Suppose that $\prod_{\alpha \in I}(X_\alpha, u_\alpha^1, u_\alpha^2)$ is a ∂-Hausdorff biclosure space. Let $\beta \in I$ and x_β, y_β be any two distinct points of X_β. Then $(x_\alpha)_{\alpha \in I}$ and $(y_\alpha)_{\alpha \in I}$ are distinct points of $\prod_{\alpha \in I} X_\alpha$. Since $\prod_{\alpha \in I}(X_\alpha, u_\alpha^1, u_\alpha^2)$ is a ∂-Hausdorff biclosure space. There exists

∂-open subset U of (X_β, u^1_β) and ∂-open subset V of (X_β, u^2_β) such that $x_\beta \in U$, $y_\beta \in V$ and $U \cap V = \emptyset$. Therefore, $(X_\beta, u^1_\beta, u^2_\beta)$ is a ∂-Hausdorff biclosure space.

Conversely, suppose that $(X_\alpha, u^1_\alpha, u^2_\alpha)$ is a ∂-Hausdorff biclosure space for each $\alpha \in I$. Let $(x_\alpha)_{\alpha\in I}$ and $(y_\alpha)_{\alpha\in I}$ be any two distinct points of $\prod\limits_{\alpha\in I} X_\alpha$. Then x_β and y_β are distinct points of X_β. Since $(X_\beta, u^1_\beta, u^2_\beta)$ is a ∂-Hausdorff biclosure space, there exists a disjoint ∂-open subset U of (X_β, u^1_β) and ∂-open subset V of (X_β, u^2_β) such that $x_\beta \in U$ and $y_\beta \in V$. Consequently, $U \times \prod\limits_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a ∂-open subset of $\prod\limits_{\alpha\in I}(X_\alpha, u^1_\alpha,)$ and $V \times \prod\limits_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a ∂-open subset of $\prod\limits_{\alpha\in I}(X_\alpha, u^2_\alpha)$ such that $(x_\alpha)_{\alpha\in I} \in U \times \prod\limits_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$, $(y_\alpha)_{\alpha\in I} \in V \times \prod\limits_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ and $(U \times \prod\limits_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha) \cap (V \times \prod\limits_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha) = \emptyset$. Hence, $\prod\limits_{\alpha\in I}(X_\alpha, u^1_\alpha, u^2_\alpha)$ is a ∂-Hausdorff biclosure space.

Proposition 3.7. *Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces. Let $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ be injective and ∂-irresolute. If (Y, v_1, v_2) is a ∂-Hausdorff biclosure space, then (X, u_1, u_2) is a ∂-Hausdorff biclosure space.*

Proof. Let x and y be any two distinct points of X. Then $f(x)$ and $f(y)$ are distinct points of Y. Since (Y, v_1, v_2) is a ∂-Hausdorff biclosure space, there exists a disjoint ∂-open subset U of (Y, v_1) and ∂-open subset V of (Y, v_2) containing $f(x)$ and $f(y)$, respectively. Since f is ∂-irresolute and $U \cap V = \emptyset$, $f^{-1}(U)$ is a ∂-open subset of (X, u_1), $f^{-1}(V)$ is a ∂-open subset of (X, u_2), $f^{-1}(U) \cap f^{-1}(V) = \emptyset$ and $x \in f^{-1}(U)$, $y \in f^{-1}(V)$. Therefore, (X, u_1, u_2) is a ∂-Hausdorff biclosure space.

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Chawalit Boonpok
Department of Mathematics
Faculty of Science
Mahasarakham University
Mahasarakham, 44150
THAILAND
email: *chawalit.b@msu.ac.th*