

THE GEODESIC CURVATURE AND GEODESIC TORSION OF THE INTERSECTION CURVE OF TWO SURFACES

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ABSTRACT. In this paper, we study the geodesic torsion and the geodesic curvature of intersection curve of two regular surfaces given by parametric-parametric and implicit-implicit equations. We express the curvature vector and the curvature by means of the geodesic curvatures of the intersection curve and the normal vectors of the surfaces.

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1. INTRODUCTION

It is well known that the curvatures of a space curve given by its parametric equation can be found easily [3, 8, 9]. But, if the space curve is given as an intersection of two surfaces in Euclidean 3-space, the curvature computations become harder. For that reason, a lot of methods were given for computing the curvatures of the intersection curve and also the curvatures (normal, mean, Gaussian) of the intersecting surfaces. Some of these methods include algorithms to compute curvatures and some consist curvature formulas [4, 5, 7].

Surfaces are generally given by their parametric or implicit equations. Because of this, surface-surface intersection problem appears commonly as parametric-parametric, parametric-implicit and implicit-implicit in which the intersection is transversal or tangential. If the normal vectors of two surfaces are not parallel at the intersecting points, the intersection is called transversal intersection. Under such an intersection, the tangent vector of the intersection curve can be computed easily by vector product of two normals.

Aléssio and Guadalupe present formulas on geodesic torsion and geodesic curvature for the intersection curve of two spacelike surfaces in Lorentz-Minkowski 3-space [2]. Hartmann, [5], provides formulas for computing the curvature and torsion of the intersection curve. Goldman derives curvature formulas for implicit curves and surfaces using the classical curvature formulas in Differential Geometry for parametric curves and surfaces [4].

Willmore gives the method for computing the Frenet apparatus of the transversal intersection curve of two implicit surfaces, [9].

Ye and Maekawa derive the Frenet vectors and curvatures of the intersection curve for all three types of transversal and tangential intersections. They obtain also the normal curvatures -in the direction of the tangent vector- of the surfaces at the intersecting points. Finally, they give algorithms for the evaluation of higher-order derivatives, [10].

Using the Implicit Function Theorem, Aléssio gives a method to compute the unit tangent, the unit principal normal and the unit binormal vectors, also the curvature and the torsion of the intersection curve of two implicit surfaces, [1].

In this study; first, after giving the method for computing the geodesic torsion and geodesic curvature of the intersection curve of two parametric surfaces, the curvature vector and the curvature of the intersection curve are expressed with the geodesic curvatures of the curve and the normal vectors of the surfaces. Later, we calculate the geodesic torsion and geodesic curvature of the intersection curve of two regular surfaces given implicitly.

2. PRELIMINARIES

Let $\alpha = \alpha(s)$ be a unit-speed curve on a regular surface in \mathbf{R}^3 . Then, $\mathbf{t} = \alpha'(s)$ is called the unit tangent vector at the point $\alpha(s)$. The vector $\mathbf{t}' = \alpha''(s)$ measures the way the curve is turning and is called the curvature vector. The length of the curvature vector denotes the curvature $\kappa(s)$ of the curve α , i.e. $\kappa(s) = \|\mathbf{t}'\|$.

Let us consider the Darboux frame $\{\mathbf{t}, \mathbf{V}, \mathbf{N}\}$ instead of the Frenet frame on the curve α , where \mathbf{N} is the surface normal restricted to α and $\mathbf{V} = \mathbf{N} \times \mathbf{t}$. Then, we have

$$\begin{cases} \mathbf{t}' &= & \kappa_g \mathbf{V} &+& \kappa_n \mathbf{N} \\ \mathbf{V}' &= & -\kappa_g \mathbf{t} &+& \tau_g \mathbf{N} \\ \mathbf{N}' &= & -\kappa_n \mathbf{t} &-& \tau_g \mathbf{V} \end{cases}, \quad (2.1)$$

where κ_n is the normal curvature of the surface in the direction of the tangent vector \mathbf{t} ; τ_g and κ_g are the geodesic torsion and geodesic curvature of the curve α , respectively, [6]. Thus, the geodesic torsion and geodesic curvature of α are given by, from (2.1),

$$\tau_g = \langle \mathbf{V}', \mathbf{N} \rangle, \quad \kappa_g = \langle \mathbf{t}', \mathbf{V} \rangle, \quad (2.2)$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product.

3. PARAMETRIC-PARAMETRIC SURFACE INTERSECTION

3.1. GEODESIC TORSION

In this section, we give the geodesic torsion of the transversal intersection curve of two surfaces.

Let A and B be two regular intersecting surfaces with the parametrizations $\mathbf{X}(u, v)$ and $\mathbf{Y}(p, q)$, respectively, i.e.

$$\mathbf{X}_u \times \mathbf{X}_v \neq \mathbf{0}, \quad \mathbf{Y}_p \times \mathbf{Y}_q \neq \mathbf{0}.$$

Then, the unit normal vectors of the surfaces A and B are given by

$$\mathbf{N}^A = \frac{\mathbf{X}_u \times \mathbf{X}_v}{\|\mathbf{X}_u \times \mathbf{X}_v\|}, \quad \mathbf{N}^B = \frac{\mathbf{Y}_p \times \mathbf{Y}_q}{\|\mathbf{Y}_p \times \mathbf{Y}_q\|}.$$

Since the normal vectors \mathbf{N}^A and \mathbf{N}^B are not parallel at the intersecting points, then the unit tangent vector of the intersection curve at a point $P = \mathbf{X}(u, v) = \mathbf{Y}(p, q)$ is given by

$$\mathbf{t} = \frac{\mathbf{N}^A \times \mathbf{N}^B}{\|\mathbf{N}^A \times \mathbf{N}^B\|}. \quad (3.1)$$

Let us denote the intersection curve of the surfaces A and B with $\alpha(s)$, where s is the arc length parameter and denote the Darboux frames of α by $\{\mathbf{t}, \mathbf{V}^A, \mathbf{N}^A\}$ and $\{\mathbf{t}, \mathbf{V}^B, \mathbf{N}^B\}$ on the surfaces A and B , respectively. Thus, from (2.2), we have

$$\tau_g^A = \langle (\mathbf{V}^A)', \mathbf{N}^A \rangle, \quad \tau_g^B = \langle (\mathbf{V}^B)', \mathbf{N}^B \rangle.$$

Since α is the intersection curve, we may also write

$$\alpha(s) = \mathbf{X}(u(s), v(s)) = \mathbf{Y}(p(s), q(s))$$

which yield

$$\alpha'(s) = \mathbf{t} = \mathbf{X}_u u' + \mathbf{X}_v v' = \mathbf{Y}_p p' + \mathbf{Y}_q q'. \quad (3.2)$$

Then, the geodesic torsion of the intersection curve α with respect to the surface A is

$$\tau_g^A = \frac{1}{\sqrt{EG - F^2}} \left\{ (EM - FL) (u')^2 + (EN - GL) u'v' + (FN - GM) (v')^2 \right\} \quad (3.3)$$

in which u', v' can be obtained by (as given by [10])

$$\begin{cases} u' &= \frac{1}{EG - F^2} (G \langle \mathbf{t}, \mathbf{X}_u \rangle - F \langle \mathbf{t}, \mathbf{X}_v \rangle) \\ v' &= \frac{1}{EG - F^2} (E \langle \mathbf{t}, \mathbf{X}_v \rangle - F \langle \mathbf{t}, \mathbf{X}_u \rangle) \end{cases}, \quad (3.4)$$

where E, F, G and L, M, N are, respectively, the first and second fundamental form coefficients of the surface A (The formula (3.3) can be found in classic books on differential geometry).

Similarly, the geodesic torsion of the intersection curve α with respect to the surface B is found from

$$\tau_g^B = \frac{1}{\sqrt{eg - f^2}} \left\{ (em - f\ell) (p')^2 + (en - g\ell) p'q' + (fn - gm) (q')^2 \right\}, \quad (3.5)$$

where e, f, g and ℓ, m, n denote, respectively, the first and second fundamental form coefficients of the surface B ; and p', q' can be computed from

$$\begin{cases} p' = \frac{1}{eg - f^2} (g\langle \mathbf{t}, \mathbf{Y}_p \rangle - f\langle \mathbf{t}, \mathbf{Y}_q \rangle) \\ q' = \frac{1}{eg - f^2} (e\langle \mathbf{t}, \mathbf{Y}_q \rangle - f\langle \mathbf{t}, \mathbf{Y}_p \rangle) \end{cases}. \quad (3.6)$$

3.2. GEODESIC CURVATURE

In this section, we are going to evaluate the geodesic curvatures of the intersection curve α according to the surfaces A and B .

The second derivative of the intersection curve α is obtained by (3.2) as

$$\begin{cases} \mathbf{t}' = \mathbf{X}_{uu} (u')^2 + 2\mathbf{X}_{uv} u'v' + \mathbf{X}_{vv} (v')^2 + \mathbf{X}_u u'' + \mathbf{X}_v v'' \\ = \mathbf{Y}_{pp} (p')^2 + 2\mathbf{Y}_{pq} p'q' + \mathbf{Y}_{qq} (q')^2 + \mathbf{Y}_p p'' + \mathbf{Y}_q q'' \end{cases}. \quad (3.7)$$

Thus, from (2.2), the geodesic curvature of the intersection curve α with respect to A is

$$\begin{aligned} \kappa_g^A = \frac{1}{\sqrt{EG - F^2}} & \left\{ \left[(F_u - \frac{E_v}{2}) \langle \mathbf{X}_u, \mathbf{t} \rangle - \frac{E_u}{2} \langle \mathbf{X}_v, \mathbf{t} \rangle \right] (u')^2 \right. \\ & + (G_u \langle \mathbf{X}_u, \mathbf{t} \rangle - E_v \langle \mathbf{X}_v, \mathbf{t} \rangle) u'v' \\ & \left. + \left[\frac{G_v}{2} \langle \mathbf{X}_u, \mathbf{t} \rangle - (F_v - \frac{G_u}{2}) \langle \mathbf{X}_v, \mathbf{t} \rangle \right] (v')^2 \right\} \\ & + \sqrt{EG - F^2} (u'v'' - v'u'') \end{aligned} \quad (3.8)$$

(The formula (3.8) can be found in classic books on differential geometry). Since u', v' are known by (3.4), we need to evaluate u'', v'' to find the geodesic curvature of the intersection curve.

From (3.7), we have

$$\mathbf{X}_u u'' + \mathbf{X}_v v'' = \mathbf{Y}_p p'' + \mathbf{Y}_q q'' + \mathbf{\Lambda}, \quad (3.9)$$

where

$$\mathbf{\Lambda} = \mathbf{Y}_{pp} (p')^2 + 2\mathbf{Y}_{pq} p'q' + \mathbf{Y}_{qq} (q')^2 - \mathbf{X}_{uu} (u')^2 - 2\mathbf{X}_{uv} u'v' - \mathbf{X}_{vv} (v')^2.$$

If we take the vector product of both hand sides of (3.9) by \mathbf{Y}_q and \mathbf{Y}_p , we obtain

$$\begin{aligned} (\mathbf{X}_u \times \mathbf{Y}_q) u'' + (\mathbf{X}_v \times \mathbf{Y}_q) v'' &= (\mathbf{Y}_p \times \mathbf{Y}_q) p'' + \mathbf{\Lambda} \times \mathbf{Y}_q \\ (\mathbf{X}_u \times \mathbf{Y}_p) u'' + (\mathbf{X}_v \times \mathbf{Y}_p) v'' &= (\mathbf{Y}_q \times \mathbf{Y}_p) q'' + \mathbf{\Lambda} \times \mathbf{Y}_p \end{aligned}$$

which yield

$$\begin{aligned} p'' &= a_{11}u'' + a_{12}v'' + a_{13} \\ q'' &= a_{21}u'' + a_{22}v'' + a_{23} \end{aligned} \quad (3.10)$$

where

$$\begin{aligned} a_{11} &= \frac{\det(\mathbf{X}_u, \mathbf{Y}_q, \mathbf{N}^B)}{\sqrt{eg-f^2}}, & a_{12} &= \frac{\det(\mathbf{X}_v, \mathbf{Y}_q, \mathbf{N}^B)}{\sqrt{eg-f^2}}, & a_{13} &= \frac{\det(\mathbf{Y}_q, \mathbf{\Lambda}, \mathbf{N}^B)}{\sqrt{eg-f^2}}, \\ a_{21} &= \frac{\det(\mathbf{Y}_p, \mathbf{X}_u, \mathbf{N}^B)}{\sqrt{eg-f^2}}, & a_{22} &= \frac{\det(\mathbf{Y}_p, \mathbf{X}_v, \mathbf{N}^B)}{\sqrt{eg-f^2}}, & a_{23} &= \frac{\det(\mathbf{\Lambda}, \mathbf{Y}_p, \mathbf{N}^B)}{\sqrt{eg-f^2}}. \end{aligned}$$

Taking the scalar product of both hand sides of (3.9) with \mathbf{N}^B yields

$$\langle \mathbf{X}_u, \mathbf{N}^B \rangle u'' + \langle \mathbf{X}_v, \mathbf{N}^B \rangle v'' = \langle \mathbf{\Lambda}, \mathbf{N}^B \rangle \quad (3.11)$$

and taking the scalar product of both hand sides of (3.7) with \mathbf{t} gives

$$\langle \mathbf{X}_u, \mathbf{t} \rangle u'' + \langle \mathbf{X}_v, \mathbf{t} \rangle v'' = -\langle \mathbf{X}_{uu}, \mathbf{t} \rangle (u')^2 - 2\langle \mathbf{X}_{uv}, \mathbf{t} \rangle u'v' - \langle \mathbf{X}_{vv}, \mathbf{t} \rangle (v')^2. \quad (3.12)$$

(3.11) and (3.12) constitute a linear system with respect to u'' and v'' which has nonvanishing coefficients determinant, i.e. $\Delta = \|\mathbf{X}_u \times \mathbf{X}_v\| \cdot \|\mathbf{N}^A \times \mathbf{N}^B\| \neq 0$. Thus, u'' and v'' can be computed by solving this linear system.

Similarly, the geodesic curvature of the intersection curve with respect to the surface B is given by

$$\begin{aligned} \kappa_g^B &= \frac{1}{\sqrt{eg-f^2}} \left\{ \left[\left(f_p - \frac{e_q}{2} \right) \langle \mathbf{Y}_p, \mathbf{t} \rangle - \frac{e_p}{2} \langle \mathbf{Y}_q, \mathbf{t} \rangle \right] (p')^2 \right. \\ &\quad + (g_p \langle \mathbf{Y}_p, \mathbf{t} \rangle - e_q \langle \mathbf{Y}_q, \mathbf{t} \rangle) p'q' \\ &\quad \left. + \left[\frac{g_q}{2} \langle \mathbf{Y}_p, \mathbf{t} \rangle - \left(f_q - \frac{g_p}{2} \right) \langle \mathbf{Y}_q, \mathbf{t} \rangle \right] (q')^2 \right\} \\ &\quad + \sqrt{eg-f^2} (p'q'' - q'p''), \end{aligned} \quad (3.13)$$

where p'' and q'' can be found from (3.10).

3.3. CURVATURE VECTOR AND CURVATURE

In this section, we express the curvature vector (hence the curvature) of the intersection curve with geodesic curvatures.

Since the curvature vector \mathbf{t}' of the intersection curve is perpendicular to the tangent vector \mathbf{t} , it can be represented as a linear combination of the normal vectors \mathbf{N}^A and \mathbf{N}^B , i.e.

$$\mathbf{t}' = \lambda \mathbf{N}^A + \mu \mathbf{N}^B. \quad (3.14)$$

Now, let us find the unknowns λ and μ . By taking the scalar product of both hand sides of (3.14) with \mathbf{V}^A and \mathbf{V}^B , we get

$$\lambda = \frac{1}{\sin \theta} \kappa_g^B, \quad \mu = -\frac{1}{\sin \theta} \kappa_g^A, \quad (3.15)$$

where θ is the angle between the normal vectors of the surfaces.

If we substitute (3.15) into (3.14), we obtain the curvature vector of the intersection curve as

$$\mathbf{t}' = \frac{1}{\sin \theta} (\kappa_g^B \mathbf{N}^A - \kappa_g^A \mathbf{N}^B). \quad (3.16)$$

Hence, we may give:

Theorem 1. *Let α be a unit speed intersection curve of two regular surfaces A and B . Then, the relation between the curvature and geodesic curvatures of the intersection curve is*

$$\kappa = \frac{1}{|\sin \theta|} \sqrt{(\kappa_g^A)^2 + (\kappa_g^B)^2 - 2\kappa_g^A \kappa_g^B \cos \theta},$$

where θ is the angle between the normal vectors.

4. IMPLICIT-IMPLICIT SURFACE INTERSECTION

In this part, we give the geodesic torsion and geodesic curvature of the intersection curve of the regular surfaces A and B which are given by their implicit functions $f(x, y, z) = 0$ and $g(x, y, z) = 0$, respectively.

Since $\nabla f = (f_x, f_y, f_z) \neq \mathbf{0}$ and $\nabla g = (g_x, g_y, g_z) \neq \mathbf{0}$, the normal vectors are given by

$$\mathbf{N}^A = \frac{\nabla f}{\|\nabla f\|}, \quad \mathbf{N}^B = \frac{\nabla g}{\|\nabla g\|}.$$

The tangent vector of the intersection curve is then given by

$$\mathbf{t} = \frac{\nabla f \times \nabla g}{\|\nabla f \times \nabla g\|}. \quad (4.1)$$

4.1. GEODESIC TORSION

Let us denote the intersection curve with $\alpha(s) = (x(s), y(s), z(s))$, where s is the arc-length parameter.

Using $\tau_g^A = \langle (\mathbf{V}^A)', \mathbf{N}^A \rangle$ and $\mathbf{V}^A = \mathbf{N}^A \times \mathbf{t}$, we get

$$\tau_g^A = \frac{1}{\|\nabla f\|^2 \cdot \|\nabla f \times \nabla g\|} \langle \|\nabla f\|^2 \nabla g - \langle \nabla f, \nabla g \rangle \nabla f, (\nabla f)' \rangle.$$

Since

$$(\nabla f)' = (f_{xx}x' + f_{xy}y' + f_{xz}z', f_{xy}x' + f_{yy}y' + f_{yz}z', f_{xz}x' + f_{yz}y' + f_{zz}z'),$$

we also have

$$\begin{aligned} \tau_g^A &= \frac{1}{\|\nabla f \times \nabla g\|} \{g_x (f_{xx}x' + f_{xy}y' + f_{xz}z') + g_y (f_{xy}x' + f_{yy}y' + f_{yz}z') \\ &\quad + g_z (f_{xz}x' + f_{yz}y' + f_{zz}z')\} - \frac{\cot \theta}{\|\nabla f\|^2} \{f_x (f_{xx}x' + f_{xy}y' + f_{xz}z') \\ &\quad + f_y (f_{xy}x' + f_{yy}y' + f_{yz}z') + f_z (f_{xz}x' + f_{yz}y' + f_{zz}z')\}, \end{aligned}$$

where θ is the angle between the normal vectors. Thus, we have the following theorem:

Theorem 2. *Let A and B be two intersecting surfaces given by the implicit functions $f = 0$, $g = 0$. Then, the geodesic torsion of the intersection curve according to A and B is given, respectively, by*

$$\tau_g^A = \left(\frac{1}{\|\nabla f \times \nabla g\|} \nabla G - \frac{\cot \theta}{\|\nabla f\|^2} \nabla F \right) \Phi T$$

and

$$\tau_g^B = \left(\frac{-1}{\|\nabla f \times \nabla g\|} \nabla F + \frac{\cot \theta}{\|\nabla g\|^2} \nabla G \right) \Psi T,$$

where

$$\begin{aligned} \nabla F &= [f_x \ f_y \ f_z], \quad \nabla G = [g_x \ g_y \ g_z], \quad T = \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}, \\ \Phi &= \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{xy} & f_{yy} & f_{yz} \\ f_{xz} & f_{yz} & f_{zz} \end{bmatrix}, \quad \Psi = \begin{bmatrix} g_{xx} & g_{xy} & g_{xz} \\ g_{xy} & g_{yy} & g_{yz} \\ g_{xz} & g_{yz} & g_{zz} \end{bmatrix}. \end{aligned}$$

4.2. GEODESIC CURVATURE

Since $\alpha'(s) = \mathbf{t} = (x', y', z')$ and $\alpha''(s) = \mathbf{t}' = (x'', y'', z'')$, the geodesic curvature of the intersection curve at the point $\alpha(s_0) = P$ with respect to the surface A is, [7],

$$\kappa_g^A = \frac{1}{\|\nabla f\|} \{(y'z'' - y''z')f_x + (z'x'' - z''x')f_y + (x'y'' - x''y')f_z\}.$$

x', y', z' are known by (4.1). Thus, we must find x'', y'', z'' to evaluate the geodesic curvature.

Using $\langle \mathbf{t}', \mathbf{t} \rangle = 0$ gives us the first linear equation with respect to x'', y'', z'' as

$$x'x'' + y'y'' + z'z'' = 0. \quad (4.2)$$

On the other hand; since $\langle \mathbf{t}, \nabla f \rangle = 0$ and $\langle \mathbf{t}, \nabla g \rangle = 0$, we have

$$\langle \mathbf{t}', \nabla f \rangle = -\langle \mathbf{t}, (\nabla f)' \rangle, \quad \langle \mathbf{t}', \nabla g \rangle = -\langle \mathbf{t}, (\nabla g)' \rangle$$

which yield the other two linear equations:

$$\begin{aligned} f_x x'' + f_y y'' + f_z z'' &= -f_{xx} (x')^2 - f_{yy} (y')^2 - f_{zz} (z')^2 \\ &\quad - 2(f_{xy} x' y' + f_{xz} x' z' + f_{yz} y' z') \end{aligned} \quad (4.3)$$

$$\begin{aligned} g_x x'' + g_y y'' + g_z z'' &= -g_{xx} (x')^2 - g_{yy} (y')^2 - g_{zz} (z')^2 \\ &\quad - 2(g_{xy} x' y' + g_{xz} x' z' + g_{yz} y' z'). \end{aligned} \quad (4.4)$$

(4.2), (4.3) and (4.4) constitute a linear system with respect to x'', y'', z'' with nonzero coefficients determinant ($\Delta = \|\nabla f \times \nabla g\| \neq 0$). Hence, solving this system completes the geodesic curvature of the intersection curve.

Similarly,

$$\kappa_g^B = \frac{1}{\|\nabla g\|} \{ (y' z'' - y'' z') g_x + (z' x'' - z'' x') g_y + (x' y'' - x'' y') g_z \}.$$

5. EXAMPLES

5.1. THE INTERSECTION OF PARAMETRIC-PARAMETRIC SURFACES

Let the surface A be the sphere given by

$$\mathbf{X}(u, v) = (\cos u \cos v, \sin u \cos v, \sin v), \quad 0 \leq u \leq \pi, \quad 0 \leq v \leq 2\pi$$

and the surface B be the cylinder given by

$$\mathbf{Y}(p, q) = \left(\frac{1}{2} \cos p + \frac{1}{2}, \frac{1}{2} \sin p, q \right), \quad -\pi \leq p \leq \pi, \quad -1.5 \leq q \leq 1.5,$$

respectively (Figure 1).

Let us find the geodesic torsions, geodesic curvatures, curvature vector and curvature of the intersection curve at the point $P = \mathbf{X} \left(\frac{\pi}{4}, \frac{\pi}{4} \right) = \mathbf{Y} \left(\frac{\pi}{2}, \frac{\sqrt{2}}{2} \right) = \left(\frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2} \right)$.

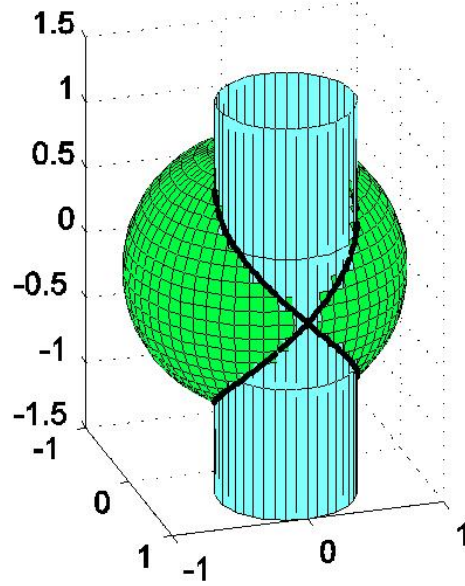


Figure 1: The intersection of the sphere and the cylinder

The partial derivatives of the surface A at P are $\mathbf{X}_u = \left(-\frac{1}{2}, \frac{1}{2}, 0\right)$, $\mathbf{X}_v = \left(-\frac{1}{2}, -\frac{1}{2}, \frac{\sqrt{2}}{2}\right)$, $\mathbf{X}_{uu} = \left(-\frac{1}{2}, -\frac{1}{2}, 0\right)$, $\mathbf{X}_{uv} = \left(\frac{1}{2}, -\frac{1}{2}, 0\right)$, $\mathbf{X}_{vv} = \left(-\frac{1}{2}, -\frac{1}{2}, -\frac{\sqrt{2}}{2}\right)$.

Thus, we get the unit normal and the first and second fundamental form coefficients of A at P as $\mathbf{N}^A = \left(\frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2}\right)$, $E = -L = \frac{1}{2}$, $G = 1$, $E_v = N = -1$, $F = E_u = F_u = F_v = G_u = G_v = M = 0$.

Similarly, for the surface B at the point P we have $\mathbf{Y}_p = \left(-\frac{1}{2}, 0, 0\right)$, $\mathbf{Y}_q = (0, 0, 1)$, $\mathbf{Y}_{pp} = \left(0, -\frac{1}{2}, 0\right)$, $\mathbf{Y}_{pq} = \mathbf{Y}_{qq} = \mathbf{0}$, $\mathbf{N}^B = (0, 1, 0)$, $e = \frac{1}{4}$, $g = 1$, $\ell = -\frac{1}{2}$, $f = e_p = e_q = f_p = f_q = g_p = g_q = m = n = 0$.

Then, the tangent vector of the intersection curve at P is $\mathbf{t} = \left(-\frac{\sqrt{6}}{3}, 0, \frac{\sqrt{3}}{3}\right)$. Using (3.4) and (3.6), we find $u' = v' = \frac{\sqrt{6}}{3}$, $p' = \frac{2\sqrt{6}}{3}$, $q' = \frac{\sqrt{3}}{3}$ at P . So, $\mathbf{\Lambda} = \left(0, 0, \frac{\sqrt{2}}{3}\right)$, $u'' = v'' = \frac{2}{9}$, $p'' = \frac{4}{9}$, $q'' = -\frac{2\sqrt{2}}{9}$ are obtained.

If we substitute these results into (3.3) and (3.5), we get the geodesic torsions as

$$\tau_g^A = 0, \quad \tau_g^B = \frac{2\sqrt{2}}{3};$$

and into (3.8) and (3.13), the geodesic curvatures as

$$\kappa_g^A = \frac{5\sqrt{3}}{9}, \quad \kappa_g^B = -\frac{2\sqrt{3}}{9}$$

of the intersection curve at the point P .

Hence, from (3.16) the curvature vector at P is $\mathbf{t}' = \left(-\frac{2}{9}, -\frac{4}{3}, -\frac{2\sqrt{2}}{9}\right)$ and then the curvature at P is $\kappa = \frac{2}{9}\sqrt{39}$.

Using the implicit functions of these surfaces and the method of Ye and Maekawa [10], Aléssio had given the above curvature vector and curvature by means of the normal curvatures of the intersection curve in the direction of \mathbf{t} , [1].

5.2. THE INTERSECTION OF IMPLICIT-IMPLICIT SURFACES

Let the surface A be the saddle surface given by $f(x, y, z) = z - xy = 0$ and the surface B be the elliptic paraboloid given by $g(x, y, z) = x^2 + y^2 + z - 3 = 0$ (Figure 2). The normal vectors of A and B at the intersection point $P = (1, -2, -2)$ are $\nabla f = (2, -1, 1)$ and $\nabla g = (2, -4, 1)$, respectively. We also have at P

$$f_{xx} = f_{xz} = f_{yy} = f_{yz} = f_{zz} = 0, \quad f_{xy} = -1$$

and

$$g_{xy} = g_{xz} = g_{yz} = g_{zz} = 0, \quad g_{xx} = g_{yy} = 2.$$

Thus, we obtain $\|\nabla f \times \nabla g\| = 3\sqrt{5}$, $\nabla F = [2 \ -1 \ 1]$, $\nabla G = [2 \ -4 \ 1]$ and

$$T = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ 0 \\ -\frac{2}{\sqrt{5}} \end{bmatrix}, \quad \Phi = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \Psi = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which yield $\tau_g^A = \frac{1}{6}$ and $\tau_g^B = -\frac{16}{105}$.

To find the geodesic curvatures of the intersection curve, we must solve the linear system obtained by (4.2), (4.3) and (4.4):

$$\begin{aligned} x'' - 2z'' &= 0 \\ 2x'' - y'' + z'' &= 0 \\ 2x'' - 4y'' + z'' &= -\frac{2}{5}. \end{aligned}$$

The solution of this system is $x'' = \frac{4}{75}$, $y'' = \frac{2}{15}$, $z'' = \frac{2}{75}$. Hence, the geodesic curvature of the intersection curve with respect to A is $\kappa_g^A = \frac{2\sqrt{30}}{75}$ and according to B is $\kappa_g^B = \frac{2\sqrt{105}}{175}$.

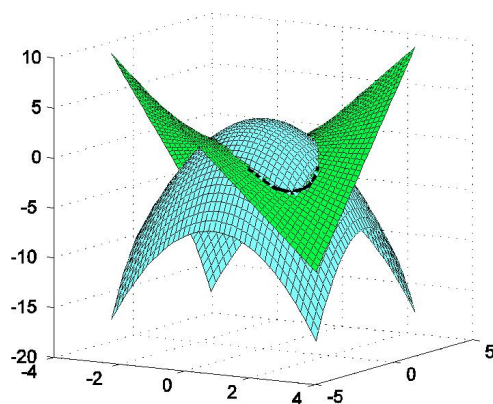


Figure 2: The intersection of the saddle surface and the elliptic paraboloid

The curvature vector of the intersection curve at P is $\mathbf{t}' = \left(\frac{4}{75}, \frac{2}{15}, \frac{2}{75}\right)$ and the curvature is $\kappa = \frac{2\sqrt{30}}{75}$.

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