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STABILITY OF QUEUEING NETWORK SYSTEM WITH TWO STATIONS AND N CLASSES

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Abstract. In this paper we study the ergodicity of the queueing network system with two stations and N classes "N is a multiple of 4" with $(\frac{N}{4}-1)$ feedbacks at the first station and $\frac{N}{4}$ feedbacks at the second one under the FIFO policy and the usual conditions $\rho_1 = m_1 + \sum_{l_1=4}^{N} m_{l_1} + \sum_{l_2=5}^{N-3} m_{l_2} < 1$ and $\rho_2 = \sum_{l_3=2}^{N-2} m_{l_3} + \sum_{l_4=3}^{N-1} m_{l_4} < 1$. By using the fluid model criterion presented by Rybko, Stolyar and Dai, we show that if $\rho_1 \leq \rho_2$ then the fluid model is stable and the stochastic queueing network system is ergodic.

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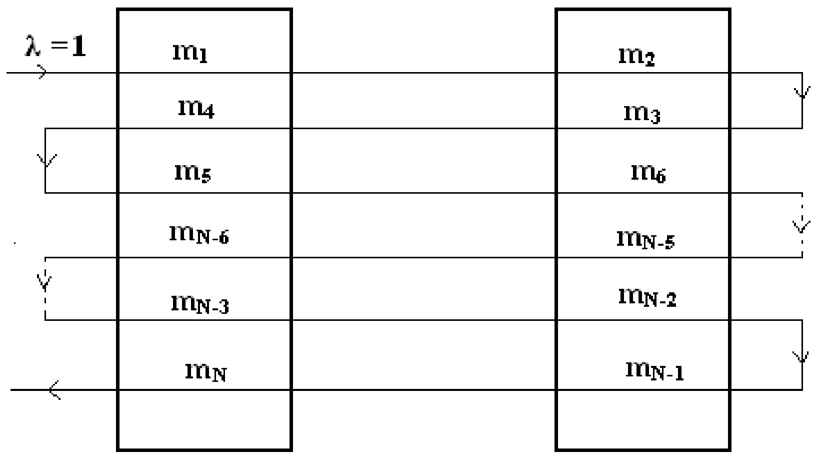
1. Introduction

Our network is composed of two queues $(i = 1, 2)$. At each queue there is one server and a waiting room of infinite capacity. Customers follow a route fixed by the network . They arrive from outside at rate 1, they will make the queue 1 where they need a service of mean m_1, then they align the second queue where they need at first a service of mean m_2, and then they test a feedback at this queue requiring a service of mean m_3, then they return to the first queue where they need a service of mean m_4 and then they test a feedback at this queue requiring a service of mean m_5, after that they will align the queue 2 where they ask once again a service of mean m_6, the customers continue their ask for services until they return definitively to the first queue where they ask for the last time for a service of mean m_N, and they leave the network. Consequently we have N classes of customers. The discipline is FIFO in the two queues.



Network with two stations and N classes

The necessary conditions of stability are

$$\rho_1 = m_1 + \sum_{l_1=4}^{N} m_{l_1} + \sum_{l_2=5}^{N-3} m_{l_2} < 1 \text{ and } \rho_2 = \sum_{l_3=2}^{N-2} m_{l_3} + \sum_{l_4=3}^{N-1} m_{l_4} < 1. \tag{1}$$

with l_1 multiple of 4, l_2 an odd number such that the difference between the l_2 equalizes to 4, l_3 an odd number such that the difference between the l_3 equalizes to 4, l_4 is an odd number, the difference between the l_4 also equalizes to 4.

2. Fluid model

General presentations

For each integer $n \geq 1$, $\tau(n)$ is the time of the inter-arrival between the arrival of the $(n-1)th$ customer and that of the nth customer from outside; the first customer arrives at time $\tau(1)$.

The times of services for the nth customer in the various classes are $\sigma_1(n), ..., \sigma_N(n)$:
We make the following assumptions on the network. At first, we have:

$$\{(\tau(n), \sigma_1(n), ..., \sigma_N(n)), n \geq 1\} \text{ is an i.i.d sequence.} \tag{2}$$

Next, we put some moment assumptions on interarrival and service times,
we assume that

$$\mathbb{E}[\tau(1)] < \infty \text{ and } \mathbb{E}[\sigma_k(1)] = m_k < \infty, \text{ for } k = 1, ..., N. \tag{3}$$

Finally, we suppose that interarrival times are unbounded and spread out, i.e.

$$\forall x > 0,\ \ \mathbb{P}[\tau(1) \geq x] > 0. \tag{4}$$

We also suppose for some integer $n > 0$ and some function $p(x) > 0$ on $\mathbb{R}_+$ with $\int_0^\infty p(x)dx > 0$,

$$\mathbb{P}[a \leq \sum_{j=1}^{n} \tau(j) \leq b] \geq \int_a^b p(x)dx, \ \text{ for any } \ 0 \leq a < b. \tag{5}$$

Without loss of general information, we suppose that $\mathbb{E}[\tau(1)] = 1$. For $i = 1, 2$, the workload for server i per unit of time is $\rho_1 = m_1 + \sum_{l_1=4}^{N} m_{l_1} + \sum_{l_2=5}^{N-3} m_{l_2}$ and $\rho_2 = \sum_{l_3=2}^{N-2} m_{l_3} + \sum_{l_4=3}^{N-1} m_{l_4} < 1$. In all this work, we suppose that conditions (1) are satisfied. We suppose that the service policy in the two stations is FIFO.
In Dai [3] or Dumas [4], authors have presented a stochastic process $\{X(t), t \geq 0\}$ which describe the dynamic of the queueing network system.
For each $t \geq 0$, $X(t) = (X_1(t), X_2(t))$ where $X_i(t)$ is the state at the station i at time t.
Since the policy utilized is FIFO we need to take

$$X_i(t) = (C_i(i, 1), ..., C_t(i, N_i(t)), u(t), v_i(t)) \tag{6}$$

where $N_i(t)$ is the number of customers at the queue i at time $t \geq 0$ and $C_t(i, l)$ is the class of l^{th} customer at the queue i at time t.
Here, $u(t)$ is the residual time for the next customer who arrives from outside and $v_i(t)$ is the residual service time of the customer being maintained at station i at time t (by convention, if $N_i(t) = 0, v_i(t) = 0$). In the presentations (2) and (3), the process $(X_t)_{t\geq 0}$ is a piecewise deterministic Markov process, see Dai [3]. As usual we identify the stability of our network by the Harris positive recurrence of $(X_t)_{t\geq 0}$.
We will use the concept of limit fluid presented by Rybko and Stolyar [5] and Dai [3]. In this effect we need some notations.

Definition 1 *For a given initial state x, and a given class k at the queue i,*
$Q_k(x, t)$ is the number of customers of class k at time t.
$A_k(x, t)$ is the number of arrivals from class k until time t. (by convention $Q_k(x, 0) = A_k(x, 0)$).
$D_k(x, t)$ is the number of departures from class k until time t (with $D_k(x, 0) = 0$.)
$T_k(x, t)$ is the spent time by the server $\sigma(k)$ to serve the customers of class k until

time t.
$Z_i(x,t)$ *is the immediate workload at the queue* i *at time* t.

All these processes are taken continuous. We define the corresponding processes of vectors Q, A, D *and* T *which are of dimension* N *and* $Z=(Z_1,Z_2)$.

3.Fluid limit and the fluid model

If x is the state of the network, we note by $|x|$ the total number of customers in the system in the state x. For each sequence of states $(x_n)_{n\geq 0}$ with $|x_n|>0$, $\forall n$, and for any process $(H(x_n,t))_{t\geq 0}$, we define $\overline{H}^n$ by

$$\forall t\geq 0,\ \ \overline{H}^n=\frac{H(x_n,|x_n|t)}{|x_n|}$$

Theorem 1 ***(Dai)*** *Let* (x_n) *an initial sequence of states with* $|x_n|\to+\infty$, *then there exists a subsequence* $(x_\phi(n))$ *such that* $(\overline{Q}^{\phi(n)},\overline{A}^{\phi(n)},\overline{D}^{\phi(n)},\overline{T}^{\phi(n)},\overline{Z}^{\phi(n)})$ *converges in distribution to the limit* (Q,A,D,T,Z). *This limit satisfied the following equations:*

$$Q_k(t)=Q_k(0)+\mu_{k-1}T_{k-1}(t)-\mu_kT_k(t)\ \ \textit{for}\ \ k=1,...N \tag{7}$$

with $\mu_k=\frac{1}{m_k}$ *for* k=1,...,N, $\mu_0=1$ *and* $T_0(t)=t$ *for* $t\geq 0$

$$Q_k(t)\geq 0\ \ \textit{for}\ \ k=1,...,N \tag{8}$$

$$D_k(t)(t)=\mu_kT_k(t)\ \ \textit{for}\ \ k=1,...,N \tag{9}$$

$$T_k(0)=0\ \ \textit{and}\ \ T_k(.)\ \ \textit{is nondecreasing for}\ \ k=1,...,N \tag{10}$$

$$\begin{cases} B_1(t)=T_1(t)+\sum_{l_1=4}^{N}T_{l_1}(t)+\sum_{l_2=5}^{N-3}T_{l_2}(t). \\ B_2(t)=\sum_{l_3=2}^{N-2}T_{l_3}(t)+\sum_{l_4=3}^{N-1}T_{l_4}(t). \end{cases} \tag{11}$$

$$Y_i(t)=t-B_i(t)\ \ \textit{is nondecreasing for}\ \ i=1,2 \tag{12}$$

$$Y_i(t)\ \ \textit{increases only at times}\ t\ \textit{such that}\ \ Z_i(t)=0,\ \ i=1,2 \tag{13}$$

$$\begin{cases} Z_1(t) = & m_1 Q_1(t) + \sum_{l_1=4}^{N} m_{l_1} Q_{l_1}(t) + \sum_{l_2=5}^{N-3} m_{l_2} Q_{l_2}(t). \\ Z_2(t) = & \sum_{l_3=2}^{N-2} m_{l_3} Q_{l_3}(t) + \sum_{l_4=3}^{N-1} m_{l_4} Q_{l_4}(t). \end{cases} \quad (14)$$

$$D_k(t + Z_i(t)) = Q_k(0) + A_k(t) \ \ for \ \ k = 1, ..., N, \ \ i = \sigma(k) \quad (15)$$

Definition 2 *Any solution to equations (7),...,(15) is called fluid model. Thus any fluid limit is a fluid model.*
For all $k = 1, ..., N$, *the functions* $t \to T_k(t)$ *and* $t \to t - T_k(t)$ *are nondecreasing and we have* $|T_k(t) - T_k(s)| \leq |t - s|$ *for all* $s, t \geq 0$, *thus there are absolutely continuous and by fluid equations all functions* $Q_k(.)$, $B_i(.)$, $Y_i(.)$, *and* $Z_i(.)$ *are absolutely continuous.*

The condition of work-conserving (13) is used in the the following formula

$$Si \ \ Z_i(t) > 0 \ \text{ for all } \ t \in [a, b], \ \text{ then } \ Y_i(a) = Y_i(b) \quad (16)$$

FIFO equation of (15) is also known in the following equivalent form:

$$D_k(t) = Q_k(0) + A_k(\tau_i(t)) \ \text{ for all } \ t \geq t_i = Z_i(0), \ \ i = \sigma(k), \quad (17)$$

with $\tau_i(t)$ is the reverse of the function $t \to t + Z_i(t)$.
In the stochastic context, $\tau_i(t)$ is the arrival time of current customer in service at station i if $Z_i(t) > 0$ and $\tau_i(t) = t$ if $Z_i(t) = 0$
In the following proposition, we are going to give the properties of the function $\tau_i(t)$, $i = 1, 2$
(For the proof and more details , see Chen et Zhang [2]).

Proposition 1 *For i=1,2, we have*

- *a)* $Z_i(\tau_i(t)) = t - \tau_i(t) \ \ for \ \ t \geq Z_i(0)$,
- *b)* $\tau_i(t)$ *is lipschitz function on* $[0, \infty[$,
- *c)* $\tau_i(t)$ *is a nondecreasing function and* $\tau_i(t) \to +\infty$ *when* $t \to +\infty$.

4.Stability result

Theorem 2 *In addition to (1), if we have:*

$$\rho_1 \leq \rho_2, \tag{18}$$

then any fluid model $Q(.)$ *satisfied* $\underset{t\to+\infty}{lim} |Q(t)| = 0$, *and thus the network is stable.*

Proof. Let $Z(t) = Z_1(t) + Z_2(t)$. Thus we have

$$\underset{t\to+\infty}{lim} |Q(t)| = 0 \Leftrightarrow \underset{t\to+\infty}{lim} Z(t) = 0$$

We rewrite the workloads at the two stations in a convenient form which enables us to employ the property of conservation (16).
We use the fluid equations (7), (9), (14) and FIFO equation (17), the workload in the two stations can be written as follows:

$$\begin{cases} Z_1(t) = m_1[Q_1(0) + A_1(t)] + \sum_{l_1=4}^{N} m_{l_1}[Q_{l_1}(0) + A_{l_1}(t)] \\ \qquad + \sum_{l_2=5}^{N-3} m_{l_2}[Q_{l_2}(0) + A_{l_2}(t)] - t + Y_1(t). \\ Z_2(t) = \sum_{l_3=2}^{N-2} m_{l_3}[Q_{l_3}(0) + A_{l_3}(t)] + \sum_{l_4=3}^{N-1} m_{l_4}[Q_{l_4}(0) + A_{l_4}(t)] \\ \qquad -t + Y_2(t). \end{cases} \tag{19}$$

All the relations in the continuation can be held for none $y \geq 0$ but only for all $t \geq T_0$ with T_0 a finite time has to be determined by the initial data. And since we study the behavior of $Z(t)$ when $t \to \infty$, we will omit to indicate the constant T_0.
The network is a re-entrant line , thus we have
$A_1(t) = t$, $A_2(t) = D_1(t)$, $A_3(t) = D_2(t)$, $A_4(t) = D_3(t)$,...,$A_{N-3}(t) = D_{N-4}(t)$, $A_{N-2}(t) = D_{N-3}(t)$, $A_{N-1}(t) = D_{N-2}(t)$, $A_N(t) = D_{N-1}(t)$.
FIFO equation(17) gives

$$A_1(t) = t \tag{20}$$

$$A_2(t) = D_1(t) = Q_1(0) + A_1(\tau_1(t)) \tag{21}$$

$$A_3(t) = D_2(t) = Q_2(0) + A_2(\tau_2(t)) \tag{22}$$

$$A_4(t) = D_3(t) = Q_3(0) + A_3(\tau_2(t)) \tag{23}$$

$$\vdots$$

$$A_{N-3}(t) = D_{N-4}(t) = Q_{N-4}(0) + A_{N-4}(\tau_1(t)) \tag{24}$$

$$A_{N-2}(t) = D_{N-3}(t) = Q_{N-3}(0) + A_{N-3}(\tau_1(t)) \tag{25}$$

$$A_{N-1}(t) = D_{N-2}(t) = Q_{N-2}(0) + A_{N-2}(\tau_2(t)) \tag{26}$$

$$A_N(t) = D_{N-1}(t) = Q_{N-1}(0) + A_{N-1}(\tau_2(t)) \tag{27}$$

By replacing t by $\tau_1(t)$ in (20), (23), (24), (27), and by $\tau_2(t)$ in (21), (22), (25), (26) we obtain

$$\begin{aligned}
&A_1(\tau_1(t)) = \tau_1(t)\\
&A_2(\tau_2(t)) = Q_1(0) + A_1(\tau_1(\tau_2(t)))\\
&A_3(\tau_2(t)) = Q_2(0) + A_2(\tau_2^{(2)}(t))\\
&A_4(\tau_1(t)) = Q_3(0) + A_3(\tau_2(\tau_1(t)))\\
&\quad\vdots\\
&\quad\vdots\\
&A_{N-3}(\tau_1(t)) = Q_{N-4}(0) + A_{N-4}(\tau_1^{(2)}(t))\\
&A_{N-2}(\tau_2(t)) = Q_{N-3}(0) + A_{N-3}(\tau_1(\tau_2(t)))\\
&A_{N-1}(\tau_2(t)) = Q_{N-2}(0) + A_{N-2}(\tau_2^{(2)}(t))\\
&A_N(\tau_1(t)) = Q_{N-1}(0) + A_{N-1}(\tau_2(\tau_1(t)))
\end{aligned}$$

We recapitulate the above equations. For all $t \geq T$ (with T a finite time)

$$\begin{aligned}
&A_1(t) = t\\
&A_2(t) = Q_1(0) + \tau_1(t)\\
&A_3(t) = Q_1(0) + Q_2(0) + \tau_1(\tau_2(t))\\
&A_4(t) = Q_1(0) + Q_2(0) + Q_3(0) + \tau_1(\tau_2^{(2)}(t))
\end{aligned}$$

$$\vdots$$

$$\vdots$$

$$A_{N-3}(t) = \sum_{l=1}^{N-4} Q_l(0) + \underbrace{\tau_1(\tau_2^{(2)}(\tau_1^{(2)}(...(\tau_2^{(2)}(\tau_1(t)))...)))}_{\frac{N-2}{2}}$$

$$A_{N-2}(t) = \sum_{l=1}^{N-3} Q_l(0) + \underbrace{\tau_1(\tau_2^{(2)}(\tau_1^{(2)}(...(\tau_2^{(2)}(\tau_1^{(2)}(t)))...)))}_{\frac{N-2}{2}}$$

$$A_{N-1}(t) = \sum_{l=1}^{N-2} Q_l(0) + \underbrace{\tau_1(\tau_2^{(2)}(\tau_1^{(2)}(...(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2(t))))...)))}_{\frac{N}{2}}$$

$$A_N(t) = \sum_{l=1}^{N-1} Q_l(0) + \underbrace{\tau_1(\tau_2^{(2)}(\tau_1^{(2)}(...(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t))))...)))}_{\frac{N}{2}}$$

We take again the four last equalities as follows:

$$A_{N-3}(t) = \sum_{l=1}^{N-4} Q_l(0) + h_1(t)$$
$$A_{N-2}(t) = \sum_{l=1}^{N-3} Q_l(0) + h_2(t)$$
$$A_{N-1}(t) = \sum_{l=1}^{N-2} Q_l(0) + h_3(t)$$
$$A_N(t) = \sum_{l=1}^{N-1} Q_l(0) + g_1(t)$$

The substitution of $A_k(t)$ on (19) pays

$$\begin{cases} Z_1(t) = c_1 - m_4[t - \tau_1(\tau_2^{(2)}(t))] - ... - m_{N-3}[t - h_1(t)] \\ \qquad -m_N[t - g_1(t)] + (\rho_1 - 1)t + Y_1(t). \\ Z_2(t) = c_2 - m_2(t - \tau_1(t)) - m_3(t - \tau_1(\tau_2(t))) - ... - m_{N-2}[t \\ \qquad -h_2(t)] - m_{N-1}[t - h_3(t)] + (\rho_2 - 1)t + Y_2(t). \end{cases}$$

where c_1 and c_2 are constants which are not depending on time .
So,

$$Y_1(t) = Z_1(t) + m_4[t - \tau_1(\tau_2^{(2)}(t))] + ... + m_{N-3}[t - h_1(t)] \\ +m_N[t - g_1(t)] - (\rho_1 - 1)t - c_1. \tag{28}$$

$$\begin{aligned} Y_2(t) = Z_2(t) + m_2(t - \tau_1(t)) + m_3(t - \tau_1(\tau_2(t))) + ... \\ + m_{N-2}[t - h_2(t)] + m_{N-1}[t - h_3(t)] - (\rho_2 - 1)t - c_2. \end{aligned} \tag{29}$$

By using the property (a) of the proposition 1 we can rewrite (28) in the following form:

$$\begin{aligned} &Y_1(t) = Z_1(t) + m_4[t - \tau_2(t) + \tau_2(t) - \tau_2^{(2)}(t) + \tau_2^{(2)}(t) - \tau_1(\tau_2^{(2)}(t))] + ... \\ &+ m_{N-3}[t - \tau_1(t) + \tau_1(t) - \tau_2(\tau_1(t)) + \tau_2(\tau_1(t)) + ... - h_1(t)] \\ &+ m_N[t - \tau_2(t) + \tau_2(t) - \tau_2^{(2)}(t) + \tau_2^{(2)}(t) - \tau_1(\tau_2^{(2)}(t)) + \tau_1(\tau_2^{(2)}(t)) \\ &+ \tau_1^{(2)}(\tau_2^{(2)}(t)) - \tau_1^{(2)}(\tau_2^{(2)}(t)) + \tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t))) - \tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t))) \\ &+ ... - g_1(t)] - (\rho_1 - 1)t - c_1. \end{aligned}$$

Thus,

$$\begin{aligned} &Y_1(t) = Z_1(t) + m_4[Z_2(\tau_2(t)) + Z_2(\tau_2^{(2)}(t)) + Z_1(\tau_1(\tau_2^{(2)}(t)))] \\ &+ ... + m_{N-3}[Z_1(\tau_1(t)) + Z_2(\tau_2(\tau_1(t))) + Z_2(\tau_2^{(2)}(\tau_1(t))) + ... \\ &+ Z_1(h_1(t))] + m_N[Z_2(\tau_2(t)) + Z_2(\tau_2^{(2)}(t)) + Z_1(\tau_1(\tau_2^{(2)}(t))) \\ &+ Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t))) + Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t)))) + Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t)))) \\ &+ ... + Z_2(\tau_2^{(2)}(\tau_1^{(2)}(...(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t))))...))) + Z_1(g_1(t))] \\ &- (\rho_1 - 1)t - c_1. \end{aligned} \tag{30}$$

and (29) in the following form

$$\begin{aligned} &Y_2(t) = Z_2(t) + m_2(t - \tau_1(t)) + m_3[t - \tau_2(t) + \tau_2(t) - \tau_1(\tau_2(t))] + ... \\ &+ m_{N-2}[t - \tau_1(t) + \tau_1(t) - \tau_1^{(2)}(t) + \tau_1^{(2)}(t) - \tau_2(\tau_1^{(2)}(t)) + \tau_2(\tau_1^{(2)}(t)) \\ &- \tau_2^{(2)}(\tau_1^{(2)}(t))) + \tau_2^{(2)}(\tau_1^{(2)}(t))) + ... - h_2(t)] + m_{N-1}[t - \tau_2(t) + \tau_2(t) \\ &- \tau_1(\tau_2(t)) + \tau_1(\tau_2(t)) - \tau_1^{(2)}(\tau_2(t)) + \tau_1^{(2)}(\tau_2(t)) - \tau_2(\tau_1^{(2)}(\tau_2(t))) \\ &+ \tau_2(\tau_1^{(2)}(\tau_2(t))) - \tau_2^{(2)}(\tau_1^{(2)}(\tau_2(t))) + \tau_2^{(2)}(\tau_1^{(2)}(\tau_2(t))) + ... - h_3(t)] \\ &- (\rho_2 - 1)t - c_2. \end{aligned}$$

So

$$\begin{aligned} &Y_2(t) = Z_2(t) + m_2 Z_1(\tau_1(t)) + m_3[Z_2(\tau_2(t)) + Z_1(\tau_1(\tau_2(t)))] \\ &+ ... + m_{N-2}[Z_1(\tau_1(t)) + Z_1(\tau_1^{(2)}(t)) + Z_2(\tau_2(\tau_1^{(2)}(t))) \\ &+ Z_2(\tau_2^{(2)}(\tau_1^{(2)}(t))) + ... + Z_1(h_2(t))] + m_{N-1}[Z_2(\tau_2(t)) \\ &+ Z_1(\tau_1(\tau_2(t))) + Z_1(\tau_1^{(2)}(\tau_2(t))) + Z_2(\tau_2(\tau_1^{(2)}(\tau_2(t)))) \\ &+ Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2(t)))) + ... + Z_1(h_3(t))] - (\rho_2 - 1)t - c_2. \end{aligned} \tag{31}$$

We will reduce this problem to the study of $Z_1(t)$.

Lemma 1 *If* $\lim_{t \to +\infty} Z_1(t) = 0$ *then* $\lim_{t \to +\infty} Z(t) = 0$.

Proof. Let t a time such that $Z_2(t) > 0$ and $a = max\{u < t, Z_2(u) = 0\}$, then $Y_2(a) = Y_2(t)$. By using the relation (29) and $Z_2(t) = 0$ we have

$$Z_2(t) + m_2Z_1(\tau_1(t)) + m_3(t - \tau_1(\tau_2(t))) + ... + m_{N-2}(t - h_2(t))$$
$$+m_{N-1}(t - h_3(t)) - (\rho_2 - 1)t - [Z_2(a) + m_2(a - \tau_1(a))$$
$$+m_3(a - \tau_1(\tau_2(a))) + ... + m_{N-2}(a - h_2(a)) + m_{N-1}(a - h_3(a))$$
$$-(\rho_2 - 1)a] = 0.$$

$Z_2(a) = 0$ because $a = max\{u < t, Z_2(u) = 0\}$ and $\tau_2(a) = a$ because $Z_2(a) = 0$

thus,

$$Z_2(t) + m_2Z_1(\tau_1(t)) + m_3(t - \tau_1(\tau_2(t))) + ... + m_{N-2}(t - h_2(t))$$
$$+m_{N-1}(t - h_3(t)) - m_2Z_1(\tau_1(a) - m_3(a - \tau_1(a)) - ...$$
$$-m_{N-2}(a - h_2(a)) - m_{N-1}(a - h_3(a)) = (\rho_2 - 1)(t - a).$$

$$Z_2(t) + m_2Z_1(\tau_1(t)) + m_3[t - \tau_1(\tau_2(t))] + ... + m_{N-2}[t - h_2(t)]$$
$$-m_{N-2}[\tau_1(a) - h_2(a)] + m_{N-1}[t - h_3(t)] - m_{N-1}[\tau_1(a)$$
$$-h_3(a)] - \rho_2Z_1(\tau_1(a)) = (\rho_2 - 1)(t - a).$$

where we still have

$$Z_2(t) + m_2Z_1(\tau_1(t)) + m_3[t - \tau_1(\tau_2(t))] + ... + m_{N-2}[t - h_2(t)$$
$$+(h_2(a) - \tau_1(a))] + m_{N-1}[t - h_3(t) + (h_3(a) - \tau_1(a))] - \rho_2Z_1(\tau_1(a))$$
$$= (\rho_2 - 1)(t - a).$$

As $\rho_2 < 1$, we have

$$Z_2(t) \leq Z_2(t) + m_2Z_1(\tau_1(t)) + m_3(t - \tau_1(\tau_2(t))) + ... + m_{N-2}[t - h_2(t)$$
$$+(h_2(a) - \tau_1(a))] + m_{N-1}[t - h_3(t) + (h_3(a) - \tau_1(a))]$$
$$< \rho_2Z_1(\tau_1(a)),$$

thus,

$$Z_2(t) < \rho_2 \sup_{\tau_2(a) \leq u \leq t} Z_1(u). \tag{32}$$

Now, to prove the stability, it is enough to prove that $\underset{t \to +\infty}{lim} Z_1(t) = 0$.

For any fluid solution $Q(.)$, we associate an increasing sequence of time $\{t_i\}$, as in Bertsimas, Gamarnik and Tsitsiklik [1] which satisfied

$$\begin{cases} in\,(t_{Nm+1},t_{Nm+2})\ Z_1(t)>0 & \text{and } Z_2(t)\geq 0\\ in\,(t_{Nm+2},t_{Nm+3})\ Z_1(t)>0 & \text{and } Z_2(t)>0\\ in\,(t_{Nm+3},t_{Nm+4})\ Z_1(t)\geq 0 & \text{and } Z_2(t)>0\\ in\,(t_{Nm+4},t_{Nm+5})\ Z_1(t)>0 & \text{and } Z_2(t)>0\\ in\,(t_{Nm+5},t_{Nm+6})\ Z_1(t)>0 & \text{and } Z_2(t)\geq 0\\ in\,(t_{Nm+6},t_{Nm+7})\ Z_1(t)>0 & \text{and } Z_2(t)>0\\ in\,(t_{Nm+7},t_{Nm+8})\ Z_1(t)\geq 0 & \text{and } Z_2(t)>0\\ in\,(t_{Nm+8},t_{Nm+9})\ Z_1(t)>0 & \text{and } Z_2(t)>0\\ \vdots & \\ \vdots & \end{cases}$$

$$\begin{cases} in\,(t_{Nm+(N-3)},t_{Nm+(N-2)})\ Z_1(t)>0 & \text{and } Z_2(t)\geq 0\\ in\,(t_{Nm+(N-2)},t_{Nm+(N-1)})\ Z_1(t)>0 & \text{and } Z_2(t)>0\\ in\,(t_{Nm+(N-1)},t_{Nm+N})\ Z_1(t)\geq 0 & \text{and } Z_2(t)>0\\ in\,(t_{Nm+N},t_{Nm+(N+1)})\ Z_1(t)>0 & \text{and } Z_2(t)>0 \end{cases}$$

and by continuity , $Z_2(t_{Nm+1}) = Z_2(t_{Nm+2}) = Z_2(t_{Nm+5}) = Z_2(t_{Nm+6}) = ... = Z_2(t_{Nm+(N-3)}) = Z_2(t_{Nm+(N-2)}) = 0$
and $Z_1(t_{Nm+3}) = Z_1(t_{Nm+4}) = Z_1(t_{Nm+7}) = Z_1(t_{Nm+8}) = ... = Z_1(t_{Nm+(N-1)}) = Z_1(t_{Nm+N}) = 0$.
The existence of the sequence $\{t_i\}$ is due to the fact that under necessary conditions of stability (1), for $i = 1, 2$, the points unit t which $Z_i(t) = 0$ is not limited.
If there exists $\delta > 0$ such that $Z(t) = 0$ for all $t \geq \delta$ for any fluid limit $Z(.)$, then $\lim\limits_{i\to+\infty} t_i \leq \delta$ and the network is stable. Else, there exists a fluid solution such that the associated sequence t_i satisfied $\lim\limits_{i\to+\infty} t_i = \infty$ and in all the remainder of the proof, we consider that we are in the second case.

To finish the proof of the theorem 2, we need the following inequalities:

$$\begin{cases} \sup\limits_{[t_{Nm+2},t_{Nm+4}]} Z_1(t) < Z_1(\tau_1(t_{Nm+2})),\\ \sup\limits_{[t_{Nm+6},t_{Nm+8}]} Z_1(t) < Z_1(\tau_1(t_{Nm+6})),\\ \vdots\\ \vdots\\ \sup\limits_{[t_{Nm+(N-3)},t_{Nm+N}]} Z_1(t) < Z_1(\tau_1(t_{Nm+(N-3)})), \end{cases} \tag{33}$$

$$\begin{cases} \sup\limits_{[t_{Nm+4}, t_{Nm+5}]} Z_1(t) < m_4[Z_2(\tau_2^{(2)}(t_{Nm+4})) + Z_1(\tau_1(\tau_2^{(2)}(t_{Nm+4})))]. \\ \sup\limits_{[t_{Nm+8}, t_{Nm+9}]} Z_1(t) < m_8[Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+8})))) \\ \qquad +Z_1(\tau_1(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+8})))))]. \\ \vdots \\ \vdots \\ \sup\limits_{[t_{Nm+N}, t_{Nm+(N+1)}]} Z_1(t) < m_N[Z_2(g_2(t_{Nm+N})) + Z_1(g_1(t_{Nm+N}))]. \end{cases} \tag{34}$$

$$\begin{cases} \sup\limits_{[t_{Nm+5}, t_{N(m+1)+(6-N)}]} Z_1(t) < m_4[Z_2(\tau_2^{(2)}(t_{8m+4})) + Z_1(\tau_1(\tau_2^{(2)}(t_{8m+4})))]. \\ \sup\limits_{[t_{Nm+9}, t_{N(m+1)+(10-N)}]} Z_1(t) < m_8[Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{8m+8})))) \\ \qquad +Z_1(\tau_1(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{8m+8})))))]. \\ \vdots \\ \vdots \\ \sup\limits_{[t_{Nm+(N+1)}, t_{N(m+1)+2}]} Z_1(t) < m_N[Z_2(g_2(t_{Nm+N})) + Z_1(g_1(t_{Nm+N})))]. \end{cases} \tag{35}$$

with $g_2(t) = \tau_2^{(2)}(\tau_1^{(2)}(...(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t))))...))$

Now, we will give the proof of the last inequality of the first equation (33). The detailed demonstration of second and third equations, (34) and (35) will be given in the appendix .

Proof of the last inequality of the equation (33): Let $t \in (t_{Nm+(N-3)}, t_{Nm+(N+1)})$, $Z_2(t) > 0$ and $Y_2(t) = Y_2(t_{Nm+(N-3)})$.
By using (29) and the fact that , $Z_2(t_{Nm+(N-3)}) = 0$, we have

$$\begin{aligned} &Z_2(t) + m_2(Z_1(\tau_1(t))) + m_3(t - \tau_1(\tau_2(t))) + ... + m_{N-2}[t - h_2(t)] \\ &+m_{N-1}[t - h_3(t)] - (\rho_2 - 1)t - [Z_2(a) + m_2(Z_1(\tau_1(a))) + m_3(a - \tau_1(\tau_2(a))) \\ &+... + m_{N-2}(t - h_2(a)) + m_{N-1}(t - h_3(a)) - (\rho_2 - 1)a] = 0. \end{aligned}$$

$Z_2(a) = 0$ because $a = max\{u < t, Z_2(u) = 0\}$ and $\tau_2(a) = a$ because $Z_2(a) = 0$

then

$$\begin{aligned}&Z_2(t)+m_2(Z_1(\tau_1(t)))+m_3(t-\tau_1(\tau_2(t)))+...+m_{N-2}[t-h_2(t)\\&+(h_2(t_{Nm+(N-3)}))-\tau_1(t_{Nm+(N-3)}))]+m_{N-1}[t-h_3(t)+(h_3(t_{Nm+2}))\\&-\tau_1(t_{Nm+2}))]-\rho_2 Z_1(\tau_1(t_{Nm+(N-3)}))=(\rho_2-1)(t-t_{Nm+(N-3)}),\end{aligned}$$

where

$$\begin{aligned}&m_2(Z_1(\tau_1(t)))+m_3(t-\tau_1(\tau_2(t)))+...+m_{N-2}[t-h_2(t)+(h_2(t_{Nm+(N-3)})\\&-\tau_1(t_{Nm+(N-3)}))]+m_{N-1}[t-h_3(t)+(h_3(t_{Nm+(N-3)})-\tau_1(t_{Nm+(N-3)}))]\\&=m_2(Z_1(\tau_1(t)))+m_3[t-\tau_1(t)+\tau_1(t)-\tau_1(\tau_2(t))]+...+m_{N-2}[t-\tau_1(t)\\&+\tau_1(t)-h_2(t)+(h_2(t_{Nm+(N-3)})-\tau_1(t_{Nm+(N-3)}))]+m_{N-1}[t-\tau_1(t)+\tau_1(t)\\&-h_3(t)+(h_3(t_{Nm+(N-3)})-\tau_1(t_{Nm+(N-3)}))]\\&=\rho_2 Z_1(\tau_1(t))+m_3[\tau_1(t)-\tau_1(\tau_2(t))]+...+m_{N-2}[\tau_1(t)-h_2(t))\\&+(h_2(t_{Nm+(N-3)})-\tau_1(t_{Nm+(N-3)}))]+m_{N-1}[\tau_1(t)-h_3(t)\\&+(h_3(t_{Nm+(N-3)})-\tau_1(t_{Nm+(N-3)}))].\end{aligned}$$

Then

$$\begin{aligned}&Z_2(t)+\rho_2 Z_1(\tau_1(t)+m_3[\tau_1(t)-\tau_1(\tau_2(t))]+...+m_{N-2}[\tau_1(t)-h_2(t)\\&+(h_2(t_{Nm+(N-3)})-\tau_1(t_{Nm+(N-3)}))]+m_{N-1}[\tau_1(t)-h_3(t)+(h_3(t_{Nm+(N-3)})\\&-\tau_1(t_{Nm+(N-3)}))]-\rho_2 Z_1(\tau_1(t_{Nm+(N-3)}))=(\rho_2-1)(t-t_{Nm+(N-3)}),\end{aligned}$$

which implies that (when $\rho_2<1$)

$$Z_1(\tau_1(t))<Z_1(\tau_1(t_{Nm+(N-3)}))\ \text{ for all }\ t\in[t_{Nm+(N-3)},t_{Nm+(N+1)}],$$

but the function $\tau_1(.)$ is continuous and strictly increasing from $[t_{Nm+(N-3)},t_{Nm+(N+1)}]$ in $[\tau_1(t_{Nm+(N-3)}),\tau_1(t_{Nm+(N+1)})]$. From where, for any $t\in[\tau_1(t_{Nm+(N-3)}),\tau_1(t_{Nm+(N+1)})]$, we have

$$Z_1(t)<Z_1(\tau_1(t_{Nm+(N-3)}))\quad\text{for}\quad\text{all}\quad t\in[\tau_1(t_{Nm+(N-3)}),\tau_1(t_{Nm+(N+1)})],$$

By definition of the sequence $\{t_i\}$, we have initially $Z_1(t_{Nm+(N-3)})>0$, thus $\tau_1(t_{Nm+(N-3)})<t_{Nm+(N-3)}$, and in the second place $Z_1(\tau_1(t_{Nm+N}))=0$, thus $t_{Nm+N}=\tau_1(t_{Nm+N})\le\tau_1(t_{Nm+(N+1)})$.
To conclude, we recapitulate all the results.
We have

$$\sup_{[t_{Nm+(N-3)},t_{Nm+N}]}Z_1(t)<Z_1(\tau_1(t_{Nm+(N-3)})),$$

and we have

$$\sup_{[t_{Nm+N},t_{N(m+1)+2}]} Z_1(t) < m_N Z_1(\tau_1(t_{Nm+(N-3)})),$$

the two inequalities imply, on the one hand ,

$$Z_1(t_{N(m+1)+2}) < m_N Z_1(\tau_1(t_{Nm+(N-3)})), \tag{36}$$

and in addition,

$$\sup_{[t_{Nm+(N-3)},t_{N(m+1)+2}]} Z_1(t) < Z_1(\tau_1(t_{Nm+(N-3)})). \tag{37}$$

The last inequality (37) is valid for any m, by replacing m by (m+1). We have

$$\sup_{[t_{N(m+1)+2},t_{N(m+2)+2}]} Z_1(t) < Z_1(\tau_1(t_{N(m+1)+2})).$$

Thus, by using (36), we have

$$Z_1(t) < m_N Z_1(\tau_1(t_{Nm+(N-3)})) \quad \text{for} \quad \text{all} \quad t \in [t_{N(m+1)+2}, t_{N(m+2)+2}]. \tag{38}$$

Now, let $S_m = t_{Nm+(N-3)}$. If $\underset{i\to+\infty}{lim}\, t_i = \infty$, then $\underset{i\to\infty}{lim}\, S_i = \infty$, and the inequality (38) implies that

$$\text{for} \quad \text{all} \quad t \in [S_m, S_{m+2}], \quad Z_1(t) < m_N(\sup_{S_{m-2}\leq u\leq S_m} Z_1(u)),$$

by iteration and because $m_N < 1$, we lead to the anticipated result.
All the other inequalities will be shown in the same manner.
We will finish this section by two numerical results.
It is supposed that the customers arrive according to a Poisson process with parameter $\lambda = 1$ and that the service time is an i.i.d sequence with an exponential distribution with parameters μ_i.
In the following tables, we give examples in simulation to illustrate the stability of the network when the condition (18) of the theorem 2 is verified .

- **Case of a network with two stations and eight classes**

μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7	μ_8	Percentage
0.1	0.2	0.2	0.2	0.3	0.1	0.3	0.2	51%
0.2	0.3	0.2	0.2	0.2	0.1	0.3	0.2	92%
0.2	0.2	0.2	0.1	0.1	0.2	0.2	0.2	100%
0.1	0.2	0.3	0.1	0.2	0.2	0.2	0.1	99%

The case of the averages $m_1 = 0.2359$, $m_2 = 0.2359$, $m_3 = 0.2359$, $m_4 = 0.0883$, $m_5 = 0.0883$, $m_6 = 0.2359$, $m_7 = 0.2359$, $m_8 = 0.2359$. The percentage is 100% i.e. all customers are served and the network is emptied (case of stability) .

- **Case of a network with two stations and twelve classes**

μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7	μ_8	μ_9	μ_{10}	μ_{11}	μ_{12}	Percentage
0.1	0.2	0.1	0.1	0.1	0.3	0.1	0.2	0.1	0.1	0.1	0.3	49%
0.1	0.2	0.1	0.1	0.1	0.2	0.1	0.1	0.1	0.2	0.1	0.1	100%
0.1	0.4	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	97%
0.2	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.2	0.3	0.2	95%

The case of the averages $m_1 = 0.0883$, $m_2 = 0.2359$, $m_3 = 0.0883$, $m_4 = 0.0883$, $m_5 = 0.0883$, $m_6 = 0.2359$, $m_7 = 0.0883$, $m_8 = 0.0883$, $m_9 = 0.0883$, $m_{10} = 0.2359$, $m_{11} = 0.0883$, $m_{12} = 0.0883$. The percentage is 100% i.e. all customers are served and the network is emptied (case of stability).

Now, in the following tables, we give examples in simulation to illustrate the instability of the network when the condition(18) of theorem 2 is not verified.

- **Case of a network with two stations and eight classes**

μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7	μ_8	Percentage
0.2	0.3	0.2	0.2	0.2	0.1	0.1	0.2	0%
0.3	0.1	0.1	0.2	0.3	0.2	0.1	0.1	0%
0.5	0.1	0.1	0.2	0.1	0.2	0.2	0.1	0%
0.2	0.1	0.2	0.3	0.1	0.2	0.1	0.3	0%

The case of the averages $m_1 = 0.2359$, $m_2 = 0.2979$, $m_3 = 0.2359$, $m_4 = 0.2359$, $m_5 = 0.2359$, $m_6 = 0.0883$, $m_7 = 0.0883$, $m_8 = 0.2359$. The percentage is 0% i.e. there is blocking in the network (case of instability) .

- **Case of a network with two stations and twelve classes**

μ_1	μ_2	μ_3	μ_4	μ_5	μ_6	μ_7	μ_8	μ_9	μ_{10}	μ_{11}	μ_{12}	Percentage
0.3	0.1	0.1	0.2	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0%
0.2	0.1	0.1	0.1	0.1	0.2	0.1	0.2	0.1	0.1	0.1	0.2	0%
0.4	0.1	0.1	0.1	0.1	0.1	0.2	0.1	0.1	0.1	0.1	0.1	0%
0.2	0.1	0.1	0.1	0.2	0.1	0.1	0.1	0.2	0.1	0.1	0.1	0%

The case of the averages $m_1 = 0.2979$, $m_2 = 0.0883$, $m_3 = 0.0883$, $m_4 = 0.2359$, $m_5 = 0.0883$, $m_6 = 0.0883$, $m_7 = 0.0883$, $m_8 = 0.0883$, $m_9 = 0.0883$, $m_{10} = 0.0883$, $m_{11} = 0.0883$, $m_{12} = 0.0883$. The percentage is 0% i.e. that there is blocking in the network (case of instability) .

5. Appendix

The proof of the inequalities (34) and (35) is done in two stages:
We give the proof of the last inequality of the equation (34)
1st Stage:
We will prove the following inequality :

$$Z_2(g_2(t_{Nm+N}))) + Z_1(g_1(t_{Nm+N})))) < Z_1(\tau_1(t_{Nm+N})). \quad (39)$$

Proof. By definition, $g_2(t)$ satisfies

$$t_{Nm+(N-3)} < g_2(t_{Nm+N}) < t_{Nm+(N+1)},$$

thus by using the conserving property (16), we clarify the relation (29) for $t = t_{Nm+(N-3)}$ and for $t = g_2(t_{Nm+N})$ the fact that

$$Y_2(t_{Nm+N}) = Y_2(g_2(t_{Nm+N})).$$

Then,
$Z_2(g_2(t_{Nm+N})) + \rho_2 Z_1(\tau_1(g_2(t_{Nm+N}))) + m_3[\tau_1(g_2(t_{Nm+N})) - \tau_1(\tau_2(g_2(t_{Nm+N})))]$
$+ ... + m_{N-2}[\tau_1(g_2(t_{Nm+N})) - h_2(g_2(t_{Nm+N})) + (h_2(t_{Nm+(N-3)}) - \tau_1(t_{Nm+(N-3)}))]$
$+ m_{N-1}[\tau_1(g_2(t_{Nm+N})) - h_3(g_2(t_{Nm+N})) + (h_3(t_{Nm+(N-3)}) - \tau_1(t_{Nm+(N-3)}))]$
$-\rho_2 Z_1(\tau_1(t_{Nm+(N-3)})) = (\rho_2 - 1)[g_2(t_{Nm+N}) - t_{Nm+(N-3)}]$.

The last expression can be written as follows :

$(\rho_2 - 1)(g_2(t_{Nm+N}) - t_{Nm+(N-3)})$
$= (\rho_2 - 1)[g_2(t_{Nm+N}) - g_3(t_{Nm+N}) + g_3(t_{Nm+N}) - t_{Nm+(N-3)}]$
$= -(\rho_2 - 1)[Z_2(g_2(t_{Nm+N}))] + (\rho_2 - 1)[g_3(t_{Nm+N}) - t_{Nm+(N-3)}]$,

with $g_3(t_{Nm+N}) = \tau_2(\tau_1^{(2)}(...(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))...)))$

thus

$$\rho_2[Z_2(g_2(t_{Nm+N})) + Z_1(g_1(t_{Nm+N})))] + m_3[\tau_1(g_2(t_{Nm+N})) - \tau_1(\tau_2(g_2(t_{Nm+N})))]$$
$$+ ... + m_{N-2}[\tau_1(g_2(t_{Nm+N})) - h_2(g_2(t_{Nm+N})) + (h_2(t_{Nm+(N-3)}) - \tau_1(t_{Nm+(N-3)}))]$$
$$+m_{N-1}[\tau_1(g_2(t_{Nm+N})) - h_3(g_2(t_{Nm+N})) + (h_3(t_{Nm+(N-3)}) - \tau_1(t_{Nm+(N-3)}))]$$
$$-\rho_2 Z_1(\tau_1(t_{Nm+(N-3)})) = (\rho_2 - 1)(g_3(t_{Nm+N}) - t_{Nm+(N-3)}),$$

thus

$$Z_2(g_2(t_{Nm+N}))) + Z_1(g_1(t_{Nm+N}))) < Z_1(\tau_1(t_{Nm+(N-3)})).$$

2nd Stage; We give the proof of the second equation (34)
1st case
If $\tau_2(t) \leq t_{Nm+N} \leq t$, then we will have thereafter:

$$t_{Nm+N} - \tau_1^{(2)}(...(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N}))))...) < t - \tau_2(g_2(t))$$
$$< t - \tau_1(g_1(t)),$$

i.e

$$Z_2(\tau_2(t_{Nm+N})) + Z_2(\tau_2^{(2)}(t_{Nm+N})) + ... + Z_1(\tau_1^{(2)}(...(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N}))))...))$$
$$< Z_2(\tau_2(t)) + Z_2(\tau_2^{(2)}(t)) + ... + Z_1(\tau_1^{(2)}(...(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t))))...)) + Z_2(g_2(t))$$
$$< Z_2(\tau_2(t)) + Z_2(\tau_2^{(2)}(t)) + ... + Z_1(\tau_1^{(2)}(...(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t))))...)) + Z_2(g_2(t))$$
$$+Z_1(g_1(t)),$$

but t still satisfies (like $Y_1(t) = Y_1(t_{Nm+N})$)

$$Z_1(t) + m_N[Z_2(\tau_2(t)) + Z_2(\tau_2^{(2)}(t)) + Z_1(\tau_1(\tau_2^{(2)}(t))) + Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t)))$$
$$+Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t)))) + Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t)))) + ... + Z_2(g_2(t)) + Z_1(g_1(t))]$$
$$-m_N[Z_2(\tau_2(t_{Nm+N})) + Z_2(\tau_2^{(2)}(t_{Nm+N})) + Z_1(\tau_1(\tau_2^{(2)}(t_{Nm+N})))$$
$$+Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N}))) + Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N}))))$$
$$+Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))) + ... + Z_2(g_2(t)) + Z_1(g_1(t_{Nm+N})))]$$
$$= (\rho_1 - 1)(t - t_{Nm+N}) < 0,$$

thus, we have necessarily

$$Z_1(t) < m_N[Z_2(g_2(t_{Nm+N})) + Z_1(g_1(t_{Nm+N}))].$$

2nd case: If $t_{Nm+N} < \tau_2(t) < t < t_{Nm+(N+1)}$, we have on the one hand,

$$Y_2(\tau_2(t)) = Y_2(\tau_2(t_{Nm+N})),$$

this implies that

$$\begin{aligned}
&Z_2(\tau_2(t)) + m_2 Z_1(\tau_1(\tau_2(t))) + m_3[Z_2(\tau_2^{(2)}(t))) + Z_1(\tau_1(\tau_2^{(2)}(t))))] + ...\\
&+m_{N-2}[Z_1(\tau_1(\tau_2(t))) + Z_1(\tau_1^{(2)}(\tau_2(t))) + Z_2(\tau_2(\tau_1^{(2)}(\tau_2(t)))) + Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2(t))))\\
&+... + Z_1(h_2(\tau_2(t)))] + m_{N-1}[Z_2(\tau_2^{(2)}(t)) + Z_1(\tau_1(\tau_2^{(2)}(t)))) + Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t))))\\
&+Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t))))) + Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t))))) + ... + Z_1(h_3(\tau_2(t)))]\\
&-Z_2(\tau_2(t_{Nm+N})) - m_2(Z_1(\tau_1(\tau_2(t_{Nm+N})))) - m_3[Z_2(\tau_2^{(2)}(t_{Nm+N})))\\
&+Z_1(\tau_1(\tau_2^{(2)}(t_{Nm+N}))))] - ... - m_{N-2}[Z_1(\tau_1(\tau_2(t_{Nm+N}))) + Z_1(\tau_1^{(2)}(\tau_2(t_{Nm+N})))\\
&+Z_2(\tau_2(\tau_1^{(2)}(\tau_2(t_{Nm+N})))) + Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2(t_{Nm+N})))) + ... + Z_1(h_2(\tau_2(t_{Nm+N})))]\\
&-m_{N-1}[Z_2(\tau_2^{(2)}(t_{Nm+N}))) + Z_1(\tau_1(\tau_2^{(2)}(t_{Nm+N})))) + Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N}))))\\
&+Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N}))))) + Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))) + ... + Z_1(h_3(\tau_2(t_{Nm+N})))]\\
&= (\rho_2 - 1)(\tau_2(t) - \tau_2(t_{Nm+N}))\\
&= (\rho_2 - 1)(\tau_2(t) - t + t + \tau_2(t) - \tau_2(t) + \tau_2^{(2)}(t) - \tau_2^{(2)}(t) + \tau_1(\tau_2^{(2)}(t))\\
&-\tau_1(\tau_2^{(2)}(t)) + ... + Z_2(\tau_2(\tau_1^{(2)}(...(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t))) - Z_2(\tau_2(\tau_1^{(2)}(...(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t)))\\
&-t_{Nm+N} + t_{Nm+N} - \tau_2(t_{Nm+N})) + \tau_2(t_{Nm+N}) - \tau_2(t_{Nm+N}) + \tau_2^{(2)}(t_{Nm+N})\\
&-\tau_2^{(2)}(t_{Nm+N}) + \tau_1(\tau_2^{(2)}(t_{Nm+N})) - \tau_1(\tau_2^{(2)}(t_{Nm+N})) + ...\\
&+Z_2(\tau_2(\tau_1^{(2)}(...(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N}))) - Z_2(\tau_2(\tau_1^{(2)}(...(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))
\end{aligned}$$

$$\begin{aligned}
&= -(\rho_2 - 1)[Z_2(\tau_2(t)) + Z_2(\tau_2^{(2)}(t)) + Z_1(\tau_1(\tau_2^{(2)}(t))) + Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t)))\\
&+Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t)))) + Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t)))) + ... + Z_2(\tau_2(\tau_1^{(2)}(...(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t)))]\\
&+(\rho_2 - 1)[t - t_{Nm+N} + \tau_2(t) - \tau_2(t_{Nm+N}) + Z_2(\tau_2(\tau_1^{(2)}(...(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))\\
&-Z_2(\tau_2(\tau_1^{(2)}(...(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t)))] + (\rho_2 - 1)[Z_2(\tau_2(t_{Nm+N})) + Z_2(\tau_2^{(2)}(t_{Nm+N}))\\
&+Z_1(\tau_1(\tau_2^{(2)}(t_{Nm+N}))) + Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N}))) + Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N}))))\\
&+Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))) + ... + Z_2(\tau_2(\tau_1^{(2)}(...(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))],
\end{aligned}$$

thus,

$$
\begin{aligned}
&\rho_2[Z_2(\tau_2(t)) + Z_2(\tau_2^{(2)}(t)) + Z_1(\tau_1(\tau_2^{(2)}(t))) + Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t)))\\
&+Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t)))) + Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t)))) + ...\\
&+Z_2(\tau_2(\tau_1^{(2)}(...(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t)))] + (\rho_2 - 1)[t - t_{Nm+N}\\
&+\tau_2(t) - \tau_2(t_{Nm+N}) + Z_2(\tau_2(\tau_1^{(2)}(...(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))\\
&-Z_2(\tau_2(\tau_1^{(2)}(...(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t)))] + m_2(Z_1(\tau_1(\tau_2(t)))) + m_3[Z_2(\tau_2^{(2)}(t)))\\
&+Z_1(\tau_1(\tau_2^{(2)}(t))))] + ... + m_{N-2}[Z_1(\tau_1(\tau_2(t))) + Z_1(\tau_1^{(2)}(\tau_2(t))) + Z_2(\tau_2(\tau_1^{(2)}(\tau_2(t))))\\
&+Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2(t)))) + ... + Z_1(h_2(\tau_2(t)))] + m_{N-1}[Z_2(\tau_2^{(2)}(t))) + Z_1(\tau_1(\tau_2^{(2)}(t))))\\
&+Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t)))) + Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t))))) + Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t)))))\\
&+... + Z_1(h_3(\tau_2(t)))] - \rho_2[Z_2(\tau_2(t_{Nm+N})) + Z_2(\tau_2^{(2)}(t_{Nm+N}))\\
&+Z_1(\tau_1(\tau_2^{(2)}(t_{Nm+N}))) + Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N}))) + Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N}))))\\
&+Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))) + ... + Z_2(\tau_2(\tau_1^{(2)}(...(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))]\\
&-m_2(Z_1(\tau_1(\tau_2(t_{Nm+N})))) - m_3[Z_2(\tau_2^{(2)}(t_{Nm+N}))) + Z_1(\tau_1(\tau_2^{(2)}(t_{Nm+N}))))]\\
&-... - m_{N-2}[Z_1(\tau_1(\tau_2(t_{Nm+N}))) + Z_1(\tau_1^{(2)}(\tau_2(t_{Nm+N}))) + Z_2(\tau_2(\tau_1^{(2)}(\tau_2(t_{Nm+N}))))\\
&+Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2(t_{Nm+N})))) + ... + Z_1(h_2(\tau_2(t_{Nm+N})))] - m_{N-1}[Z_2(\tau_2^{(2)}(t_{Nm+N})))\\
&+Z_1(\tau_1(\tau_2^{(2)}(t_{Nm+N})))) + Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))) + Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))))\\
&+Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N}))))) + ... + Z_1(h_3(\tau_2(t_{Nm+N})))]\\
&= (\rho_2 - 1)[t - t_{Nm+N} + \tau_2(t) - \tau_2(t_{Nm+N}) + Z_2(\tau_2(\tau_1^{(2)}(...(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))\\
&-Z_2(\tau_2(\tau_1^{(2)}(...(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t)))] + [Z_2(\tau_2^{(2)}(t)) + Z_1(\tau_1(\tau_2^{(2)}(t))) + Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t)))\\
&+Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t)))) + Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t)))) + ... + Z_2(\tau_2(\tau_1^{(2)}(...(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t)))]\\
&-[Z_2(\tau_2^{(2)}(t_{Nm+N})) + Z_1(\tau_1(\tau_2^{(2)}(t_{Nm+N}))) + Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))\\
&+Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))) + Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))) + ...\\
&+Z_2(\tau_2(\tau_1^{(2)}(...(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))].
\end{aligned}
\tag{40}
$$

with

$$
\begin{aligned}
L &= Z_2(\tau_2(t)) + Z_2(\tau_2^{(2)}(t)) + Z_1(\tau_1(\tau_2^{(2)}(t))) + Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t)))\\
&+Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t)))) + Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t)))) + ...\\
&+Z_2(\tau_2(\tau_1^{(2)}(...(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t)))
\end{aligned}
$$

and

$$
\begin{aligned}
L' &= Z_2(\tau_2(t_{Nm+N})) + Z_2(\tau_2^{(2)}(t_{Nm+N})) + Z_1(\tau_1(\tau_2^{(2)}(t_{Nm+N})))\\
&+Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N}))) + Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N}))))\\
&+Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))) + ... + Z_2(\tau_2(\tau_1^{(2)}(...(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))
\end{aligned}
$$

Like $\rho_2 = \sum_{l_3=2}^{N-2} m_{l_3} + \sum_{l_4=3}^{N-1} m_{l_4}$, the last equality can be written as follows :

$$
\begin{aligned}
&m_2[Z_2(\tau_2(t)) + Z_2(\tau_2^{(2)}(t)) + Z_1(\tau_1(\tau_2^{(2)}(t))) + Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t)))\\
&+Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t)))) + Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t)))) + ...\\
&+Z_2(\tau_2(\tau_1^{(2)}(...(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t)))] + m_2(Z_1(\tau_1(\tau_2(t))))\\
&+m_3[Z_2(\tau_2(t)) + Z_2(\tau_2^{(2)}(t))) + Z_1(\tau_1(\tau_2^{(2)}(t)))) + L]\\
&+... + m_{N-2}[Z_2(\tau_2(t)) + Z_1(\tau_1(\tau_2(t))) + Z_1(\tau_1^{(2)}(\tau_2(t)))\\
&+Z_2(\tau_2(\tau_1^{(2)}(\tau_2(t)))) + Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2(t)))) + ... + Z_1(h_2(\tau_2(t))) + L]\\
&+m_{N-1}[Z_2(\tau_2(t)) + Z_2(\tau_2^{(2)}(t))) + Z_1(\tau_1(\tau_2^{(2)}(t)))) + Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t))))\\
&+Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t))))) + Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t))))) + ... + Z_1(h_3(\tau_2(t))) + L]\\
&-m_2[Z_2(\tau_2(t_{Nm+N})) + Z_2(\tau_2^{(2)}(t_{Nm+N})) + Z_1(\tau_1(\tau_2^{(2)}(t_{Nm+N})))\\
&+Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N}))) + Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))) + Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N}))))\\
&+... + Z_2(\tau_2(\tau_1^{(2)}(...(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))] - m_2(Z_1(\tau_1(\tau_2(t_{Nm+N}))))\\
&-m_3[Z_2(\tau_2(t_{Nm+N})) + Z_2(\tau_2^{(2)}(t_{Nm+N}))) + Z_1(\tau_1(\tau_2^{(2)}(t_{Nm+N})))) + L']\\
&-... - m_{N-2}[Z_2(\tau_2(t_{Nm+N})) + Z_1(\tau_1(\tau_2(t_{Nm+N}))) + Z_1(\tau_1^{(2)}(\tau_2(t_{Nm+N})))\\
&+Z_2(\tau_2(\tau_1^{(2)}(\tau_2(t_{Nm+N})))) + Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2(t_{Nm+N}))))\\
&+... + Z_1(h_2(\tau_2(t_{Nm+N}))) + L'] - m_{N-1}[Z_2(\tau_2(t_{Nm+N})) + Z_2(\tau_2^{(2)}(t_{Nm+N})))\\
&+Z_1(\tau_1(\tau_2^{(2)}(t_{Nm+N})))) + Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))) + Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))))\\
&+Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N}))))) + ... + Z_1(h_3(\tau_2(t_{Nm+N})) + L']\\
&= (\rho_2 - 1)[t - t_{Nm+N} + \tau_2(t) - \tau_2(t_{Nm+N}) + Z_2(\tau_2(\tau_1^{(2)}(...(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))\\
&-Z_2(\tau_2(\tau_1^{(2)}(...(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t)))] + [Z_2(\tau_2^{(2)}(t)) + Z_1(\tau_1(\tau_2^{(2)}(t)))\\
&+Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t))) + Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t)))) + Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t))))\\
&+... + Z_2(\tau_2(\tau_1^{(2)}(...(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t)))] - [Z_2(\tau_2^{(2)}(t_{Nm+N})) + Z_1(\tau_1(\tau_2^{(2)}(t_{Nm+N})))\\
&+Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N}))) + Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))) + Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))) + ...\\
&+Z_2(\tau_2(\tau_1^{(2)}(...(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))].
\end{aligned}
$$

In addition, we have $Y_1(t) = Y_1(t_{Nm+N})$, then the relation (30) gives

$$
\begin{aligned}
&Z_1(t) + m_N[Z_2(\tau_2(t)) + Z_2(\tau_2^{(2)}(t)) + Z_1(\tau_1(\tau_2^{(2)}(t)))\\
&+Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t))) + Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t)))) + Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t))))\\
&+... + Z_2(g_2(t)) + Z_1(g_1(t))] - m_N[Z_2(\tau_2(t_{Nm+N})) + Z_2(\tau_2^{(2)}(t_{Nm+N}))\\
&+Z_1(\tau_1(\tau_2^{(2)}(t_{Nm+N}))) + Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))\\
&+Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))) + Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))) + ...\\
&+Z_2(g_2(t)) + Z_1(g_1(t_{Nm+N})))] = (\rho_1 - 1)(t - t_{Nm+N}).
\end{aligned}
\tag{41}
$$

which can be written as follows :

$$
\begin{aligned}
&Z_1(t) + m_N[Z_2(\tau_2(t)) + Z_2(\tau_2^{(2)}(t)) + Z_1(\tau_1(\tau_2^{(2)}(t))) + Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t)))\\
&+Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t)))) + Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t)))) + ... + Z_2(g_2(t)) + Z_1(g_1(t))]\\
&-(\rho_1 - 1)(t - t_{Nm+N})\\
&= m_N[Z_2(\tau_2(t_{Nm+N})) + Z_2(\tau_2^{(2)}(t_{Nm+N})) + Z_1(\tau_1(\tau_2^{(2)}(t_{Nm+N})))\\
&+Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N}))) + Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N}))))\\
&+Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))) + ... + Z_2(\tau_2(\tau_1^{(2)}(...(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))]\\
&+m_N[Z_2(g_2(t_{Nm+N})) + Z_1(g_1(t_{Nm+N})))].
\end{aligned}
$$

thus, if

$$
\begin{aligned}
&Z_1(t) \geq m_N[Z_2(g_2(t_{Nm+N})) + Z_1(g_1(t_{Nm+N})))].
\end{aligned}
$$

then,

$$
\begin{aligned}
&m_N[Z_2(\tau_2(t)) + Z_2(\tau_2^{(2)}(t)) + Z_1(\tau_1(\tau_2^{(2)}(t))) + Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t)))\\
&+Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t)))) + Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t)))) + ... + Z_2(g_2(t))Z_1(g_1(t))]\\
&-(\rho_1 - 1)(t - t_{Nm+N}) \leq m_N[Z_2(\tau_2(t_{Nm+N})) + Z_2(\tau_2^{(2)}(t_{Nm+N}))\\
&+Z_1(\tau_1(\tau_2^{(2)}(t_{Nm+N}))) + Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))\\
&+Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))) + Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))) + ...\\
&+Z_2(\tau_2(\tau_1^{(2)}(...(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))].
\end{aligned}
$$

by using the property of the function $\tau_i(.)$, for $i = 1, 2$, we have

$$
\begin{aligned}
&[Z_2(\tau_2(t)) + Z_2(\tau_2^{(2)}(t)) + Z_1(\tau_1(\tau_2^{(2)}(t))) + Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t)))\\
&+Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t)))) + Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t)))) + ... + Z_1(\tau_1(h_1(t)))]
\end{aligned}
$$

$$= [t - \tau_1(h_1(t))]$$

$$> [t - \tau_1(\tau_2(\tau_1(...(\tau_2(\tau_1(\tau_2(t))))...)))]$$

$$
\begin{aligned}
&= [Z_2(\tau_2(t)) + Z_1(\tau_1(\tau_2(t))) + Z_2(\tau_2(\tau_1(\tau_2(t)))) + Z_1(\tau_1(\tau_2(\tau_1(\tau_2(t)))))\\
&+... + Z_1(g_4(t))].
\end{aligned}
$$

with $g_4 = \tau_1(\tau_2(\tau_1(...(\tau_2(\tau_1(\tau_2(t))))...)))$

thus,

$$
\begin{aligned}
&(\rho_1-1)(t-t_{Nm+N})+m_N[Z_2(\tau_2(t_{Nm+N}))+Z_2(\tau_2^{(2)}(t_{Nm+N})))\\
&+Z_1(\tau_1(\tau_2^{(2)}(t_{Nm+N}))+Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))+Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N}))))\\
&+Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N}))))+...+Z_2(\tau_2(\tau_1^{(2)}(...(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))]\\
&-m_N[Z_2(\tau_2(t))+Z_1(\tau_1(\tau_2(t)))+Z_2(\tau_2(\tau_1(\tau_2(t))))+Z_1(\tau_1(\tau_2(\tau_1(\tau_2(t)))))\\
&+...+Z_1(g_4(t))))]>0
\end{aligned}
\tag{42}
$$

The relation (40) allows us to write

$$
\begin{aligned}
&\rho_2[Z_2(\tau_2(t_{Nm+N}))+Z_2(\tau_2^{(2)}(t_{Nm+N}))+Z_1(\tau_1(\tau_2^{(2)}(t_{Nm+N})))\\
&+Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))+Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N}))))\\
&+Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N}))))+...+Z_2(\tau_2(\tau_1^{(2)}(...(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))]\\
&-\rho_2[Z_2(\tau_2(t))+Z_2(\tau_2^{(2)}(t))+Z_1(\tau_1(\tau_2^{(2)}(t)))+Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t)))\\
&+Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t))))+Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t))))+...+Z_1(g_1(t)))]\\
&<(1-\rho_2)[t-t_{Nm+N}+\tau_2(t)-\tau_2(t_{Nm+N})+Z_2(\tau_2(\tau_1^{(2)}(...(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))\\
&-Z_2(\tau_2(\tau_1^{(2)}(...(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t)))]+[Z_2(\tau_2^{(2)}(t_{Nm+N}))+Z_1(\tau_1(\tau_2^{(2)}(t_{Nm+N})))\\
&+Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))+Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N}))))+Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N}))))\\
&+...+Z_2(\tau_2(\tau_1^{(2)}(...(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))]-[Z_2(\tau_2^{(2)}(t))\\
&+Z_1(\tau_1(\tau_2^{(2)}(t)))+Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t)))+Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t))))+Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t))))\\
&+...+Z_2(\tau_2(\tau_1^{(2)}(...(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t)))]-m_2(Z_1(\tau_1(\tau_2(t_{Nm+N}))))\\
&-m_3[Z_2(\tau_2(t_{Nm+N}))+Z_2(\tau_2^{(2)}(t_{Nm+N})))+Z_1(\tau_1(\tau_2^{(2)}(t_{Nm+N}))))]\\
&-...-m_{N-2}[Z_2(\tau_2(t_{Nm+N}))+Z_1(\tau_1(\tau_2(t_{Nm+N})))+Z_1(\tau_1^{(2)}(\tau_2(t_{Nm+N})))\\
&+Z_2(\tau_2(\tau_1^{(2)}(\tau_2(t_{Nm+N}))))+Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2(t_{Nm+N}))))+...+Z_1(h_2(\tau_2(t_{Nm+N})))]\\
&-m_{N-1}[Z_2(\tau_2(t_{Nm+N}))+Z_2(\tau_2^{(2)}(t_{Nm+N})))+Z_1(\tau_1(\tau_2^{(2)}(t_{Nm+N}))))\\
&+Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N}))))+Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))))+Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))))\\
&+...+Z_1(h_3(\tau_2(t_{Nm+N})))]
\end{aligned}
\tag{43}
$$

In addition the relation (41) is rewritten as follows

$$
\begin{aligned}
&Z_1(t)+m_N[Z_2(h_1(t)))+Z_1(\tau_1(h_1(t))))]+(1-\rho_1)(t-t_{Nm+N})\\
&=m_N[Z_2(\tau_2(t_{Nm+N}))+Z_2(\tau_2^{(2)}(t_{Nm+N}))+Z_1(\tau_1(\tau_2^{(2)}(t_{Nm+N})))\\
&+Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))+Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N}))))\\
&+Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N}))))+...+Z_2(\tau_2(\tau_1^{(2)}(...(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))]\\
&+m_N[Z_2(\tau_2(t))+Z_2(\tau_2^{(2)}(t))+Z_1(\tau_1(\tau_2^{(2)}(t)))+Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t)))\\
&+Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t))))+Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t))))+...+Z_1(\tau_1(h_1(t)))]
\end{aligned}
\tag{44}
$$

Now, one will distinguish two cases according to the right-hand side term from the equality (44) is positive or not. If

$$\begin{aligned}
&Z_2(\tau_2(t_{Nm+N})) + Z_2(\tau_2^{(2)}(t_{Nm+N})) + Z_1(\tau_1(\tau_2^{(2)}(t_{Nm+N})))\\
&+Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N}))) + Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N}))))\\
&+Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))) + ... + Z_2(\tau_2(\tau_1^{(2)}(...(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))\\
&\leq Z_2(\tau_2(t)) + Z_2(\tau_2^{(2)}(t)) + Z_1(\tau_1(\tau_2^{(2)}(t))) + Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t)))\\
&+Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t)))) + Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t)))) + ... + Z_1(g_1(t)))
\end{aligned}$$

Then according to the equality (44) and like ($\rho_1 < 1$) we have

$$Z_1(t) < m_N[Z_2(g_2(t)) + Z_1(g_1(t)))]$$

Which is the required result, if not, we have

$$\begin{aligned}
&Z_2(\tau_2(t_{Nm+N})) + Z_2(\tau_2^{(2)}(t_{Nm+N})) + Z_1(\tau_1(\tau_2^{(2)}(t_{Nm+N})))\\
&+Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N}))) + Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N}))))\\
&+Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))) + ... + Z_2(\tau_2(\tau_1^{(2)}(...(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N}))) >\\
&Z_2(\tau_2(t)) + Z_2(\tau_2^{(2)}(t)) + Z_1(\tau_1(\tau_2^{(2)}(t))) + Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t)))\\
&+Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t)))) + Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t)))) + ... + Z_1(g_1(t))).
\end{aligned}$$

Then since $m_N < \rho_1 \leq \rho_2$, we have

$$\begin{aligned}
&m_N[Z_2(\tau_2(t_{Nm+N})) + Z_2(\tau_2^{(2)}(t_{Nm+N})) + Z_1(\tau_1(\tau_2^{(2)}(t_{Nm+N})))\\
&+Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N}))) + Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N}))))\\
&+Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))) + ... + Z_2(\tau_2(\tau_1^{(2)}(...(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))]\\
&-m_N[Z_2(\tau_2(t)) + Z_2(\tau_2^{(2)}(t)) + Z_1(\tau_1(\tau_2^{(2)}(t))) + Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t)))\\
&+Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t)))) + Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t)))) + ... + Z_1(g_1(t)))]\\
&< \rho_2[Z_2(\tau_2(t_{Nm+N})) + Z_2(\tau_2^{(2)}(t_{Nm+N})) + Z_1(\tau_1(\tau_2^{(2)}(t_{Nm+N})))\\
&+Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N}))) + Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N}))))\\
&+Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))) + ... + Z_2(\tau_2(\tau_1^{(2)}(...(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))]\\
&-\rho_2[Z_2(\tau_2(t)) + Z_2(\tau_2^{(2)}(t)) + Z_1(\tau_1(\tau_2^{(2)}(t))) + Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t)))\\
&+Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t)))) + Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t)))) + ... + Z_1(g_1(t)))]
\end{aligned}$$

By using always the equality (44), we obtain

$$
\begin{aligned}
&Z_1(t) - m_N[Z_2(g_2(t_{Nm+N})) + Z_1(g_1(t_{Nm+N})))] < \rho_2[Z_2(\tau_2(t_{Nm+N}))\\
&+Z_2(\tau_2^{(2)}(t_{Nm+N})) + Z_1(\tau_1(\tau_2^{(2)}(t_{Nm+N}))) + Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))\\
&+Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))) + Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))) + ...\\
&+Z_2(\tau_2(\tau_1^{(2)}(...(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))] - \rho_2[Z_2(\tau_2(t)) + Z_2(\tau_2^{(2)}(t))\\
&+Z_1(\tau_1(\tau_2^{(2)}(t))) + Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t))) + Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t))))\\
&+Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t)))) + ... + Z_1(g_1(t)))]
\end{aligned}
$$

this implies, according to (43),

$$
\begin{aligned}
&Z_1(t) - m_N[Z_2(g_2(t_{Nm+N})) + Z_1(g_1(t_{Nm+N})))] + (1-\rho_1)(t - t_{Nm+N})\\
&< (1-\rho_2)[t - t_{Nm+N} + \tau_2(t) - \tau_2(t_{Nm+N}) + Z_2(\tau_2(\tau_1^{(2)}(...(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))\\
&-Z_2(\tau_2(\tau_1^{(2)}(...(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t)))] + [Z_2(\tau_2^{(2)}(t_{Nm+N})) + Z_1(\tau_1(\tau_2^{(2)}(t_{Nm+N})))\\
&+Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N}))) + Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))) + Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N}))))\\
&+... + Z_2(\tau_2(\tau_1^{(2)}(...(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))] - m_2(Z_1(\tau_1(\tau_2(t_{Nm+N}))))\\
&-m_3[Z_2(\tau_2^{(2)}(t_{Nm+N}))) + Z_1(\tau_1(\tau_2^{(2)}(t_{Nm+N}))))] - ... - m_{N-2}[Z_1(\tau_1(\tau_2(t_{Nm+N})))\\
&+Z_1(\tau_1^{(2)}(\tau_2(t_{Nm+N}))) + Z_2(\tau_2(\tau_1^{(2)}(\tau_2(t_{Nm+N})))) + Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2(t_{Nm+N}))))\\
&+... + Z_1(h_2(\tau_2(t_{Nm+N})))] - m_{N-1}[Z_2(\tau_2^{(2)}(t_{Nm+N}))) + Z_1(\tau_1(\tau_2^{(2)}(t_{Nm+N}))))\\
&+Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))) + Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N}))))) + Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))))\\
&+... + Z_1(h_3(\tau_2(t_{Nm+N})))]
\end{aligned}
$$

however $(1-\rho_2) \leq (1-\rho_1)$, then,

$$
\begin{aligned}
&Z_1(t) - m_N[Z_2(g_2(t_{Nm+N})) + Z_1(g_1(t_{Nm+N})))] < -m_2(Z_1(\tau_1(\tau_2(t_{Nm+N}))))\\
&-m_3[Z_2(\tau_2^{(2)}(t_{Nm+N}))) + Z_1(\tau_1(\tau_2^{(2)}(t_{Nm+N}))))] - ... - m_{N-2}[Z_1(\tau_1(\tau_2(t_{Nm+N})))\\
&+Z_1(\tau_1^{(2)}(\tau_2(t_{Nm+N}))) + Z_2(\tau_2(\tau_1^{(2)}(\tau_2(t_{Nm+N})))) + Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2(t_{Nm+N}))))\\
&+... + Z_1(h_2(\tau_2(t_{Nm+N})))] - m_{N-1}[Z_2(\tau_2^{(2)}(t_{Nm+N}))) + Z_1(\tau_1(\tau_2^{(2)}(t_{Nm+N}))))\\
&+Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))) + Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N}))))) + Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))))\\
&+... + Z_1(h_3(\tau_2(t_{Nm+N})))] + [Z_2(\tau_2^{(2)}(t_{Nm+N})) + Z_1(\tau_1(\tau_2^{(2)}(t_{Nm+N})))\\
&+Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N}))) + Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))) + Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N}))))\\
&+... + Z_2(\tau_2(\tau_1^{(2)}(...(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t_{Nm+N})))] < 0
\end{aligned}
$$

Which completes the proof of the last inequality of the equation (34).

The demonstration of the other inequalities will be made in the same manner.

Now, we give the proof of the inequality (35)

So, as in the previous cases it is sufficient to give only the proof of the last inequality of the equation (35).

Proof. $\forall t \in [t_{Nm+(N+1)}, t_{N(m+1)+2}]$, we have

$$Z_1(t) > 0 \ \text{ and } \ Z_2(t) \geq 0$$

We can then write $[t_{Nm+(N+1)}, t_{N(m+1)+2}] = \bigcup_{i=0}^{i=M}(a_i, a_{i+1})$ such that, at each interval (a_i, a_{i+1}) we have $\forall t \in (a_i, a_{i+1})$, $Z_1(t) > 0$ and $Z_2(t) \geq 0$ or $\forall t \in (a_i, a_{i+1})$, $Z_1(t) > 0$ and $Z_2(t) = 0$
Thus in the next we distinguish two cases:
1st case: Let $[a, b] \subset (t_{N(m+1)+1}, t_{N(m+1)+2})$ such that

$$Z_2(a) = Z_2(b) = 0 \ \text{ and } \ Z_2(t) > 0 \ \text{ for all } \ a < t < b$$

Let $t \in (a, b)$, thus $a < \tau_2(t) < t < b$ and $Y_2(\tau_2(t)) = Y_2(a)$, thus by using the relation (31) on the interval (a, $\tau_2(t)$), we have

$$\begin{aligned}
&Z_2(\tau_2(t)) + m_2(Z_1(\tau_1(\tau_2(t)))) + m_3[Z_2(\tau_2^{(2)}(t))) + Z_1(\tau_1(\tau_2^{(2)}(t))))] \\
&+ ... + m_{N-2}[Z_1(\tau_1(\tau_2(t))) + Z_1(\tau_1^{(2)}(\tau_2(t))) + Z_2(\tau_2(\tau_1^{(2)}(\tau_2(t)))) \\
&+ Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2(t)))) + ... + Z_1(h_2(\tau_2(t)))] + m_{N-1}[Z_2(\tau_2^{(2)}(t))) \\
&+ Z_1(\tau_1(\tau_2^{(2)}(t)))) + Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t)))) + Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t))))) \\
&+ Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t))))) + ... + Z_1(h_3(\tau_2(t)))] - \rho_2 Z_1(\tau_1(a)) \\
&= (\rho_2 - 1)(\tau_2(t) - a) \\
&= (\rho_2 - 1)(\tau_2(t) - t + t - a) \\
&= -(\rho_2 - 1)Z_2(\tau_2(t)) + (\rho_2 - 1)(t - a)
\end{aligned} \tag{45}$$

by using the fact that $\rho_2 = \sum_{l_3=2}^{N-2} m_{l_3} + \sum_{l_4=3}^{N-1} m_{l_4}$, it follows that

$$\begin{aligned}
&Z_2(\tau_2(t)) + m_2 Z_1(\tau_1(\tau_2(t))) + m_3[Z_2(\tau_2^{(2)}(t)) + Z_1(\tau_1(\tau_2^{(2)}(t)))] \\
&+ ... + m_{N-2}[Z_1(\tau_1(\tau_2(t))) + Z_1(\tau_1^{(2)}(\tau_2(t))) + Z_2(\tau_2(\tau_1^{(2)}(\tau_2(t)))) \\
&+ Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2(t)))) + ... + Z_1(h_2(\tau_2(t)))] + m_{N-1}[Z_2(\tau_2^{(2)}(t)) \\
&+ Z_1(\tau_1(\tau_2^{(2)}(t))) + Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t))) + Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t)))) \\
&+ Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t)))) + ... + Z_1(h_3(\tau_2(t)))] \\
&-(\sum_{l_3=2}^{N-2} m_{l_3} + \sum_{l_4=3}^{N-1} m_{l_4}) Z_1(\tau_1(a)) \\
&= -(\sum_{l_3=2}^{N-2} m_{l_3} + \sum_{l_4=3}^{N-1} m_{l_4} - 1) Z_2(\tau_2(t)) + (\rho_2 - 1)(t - a)
\end{aligned}$$

$$m_3[Z_2(\tau_2(t)) + Z_2(\tau_2^{(2)}(t)) + Z_1(\tau_1(\tau_2^{(2)}(t)))] + ... + m_{N-2}[Z_2(\tau_2(t))$$
$$+Z_1(\tau_1(\tau_2(t))) + Z_1(\tau_1^{(2)}(\tau_2(t))) + Z_2(\tau_2(\tau_1^{(2)}(\tau_2(t)))) + Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2(t))))$$
$$+... + Z_1(h_2(\tau_2(t)))] + m_{N-1}[Z_2(\tau_2(t)) + Z_2(\tau_2^{(2)}(t)) + Z_1(\tau_1(\tau_2^{(2)}(t)))$$
$$+Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t))) + Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t)))) + Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t))))$$
$$+... + Z_1(h_3(\tau_2(t)))] - (\sum_{l_3=6}^{N-2} m_{l_3} + \sum_{l_4=3}^{N-1} m_{l_4})Z_1(\tau_1(a))$$
$$= (\rho_2 - 1)(t-a) + m_2 Z_1(\tau_1(a)) - m_2(Z_2(\tau_2(t)) + Z_1(\tau_1(\tau_2(t)))$$

This last relation is a consequence of the property of conservation appliqued at the second station. By using the same property for the first station on the interval (a,t), $Y_1(t) = Y_1(a)$ and the relation (30) implies this

$$\begin{aligned} &Z_1(t) + m_N[Z_2(\tau_2(t)) + Z_2(\tau_2^{(2)}(t)) + Z_1(\tau_1(\tau_2^{(2)}(t))) \\ &+Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t))) + Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t)))) + Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t)))) \\ &+... + Z_1(g_1(t))] - Z_1(a) - m_N Z_1(\tau_1(a)) = (\rho_1 - 1)(t-a) \end{aligned} \quad (46)$$

(because the fact that $Z_2(a) = Z_2(b) = 0$ involves that $\tau_2(a) = \tau_2^{(2)}(a) = a$)

This equality implies that

$$Z_1(t) < Z_1(a),$$

in other words, the last equality implies, on the one hand that

$$\begin{aligned} &m_N[Z_2(\tau_2(t)) + Z_2(\tau_2^{(2)}(t)) + Z_1(\tau_1(\tau_2^{(2)}(t))) + Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t))) \\ &+Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t)))) + Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t)))) + ... + Z_1(g_1(t)))] \\ &< m_N Z_1(\tau_1(a)) \end{aligned} \quad (47)$$

and in addition, it can be written as follows:
$(\rho_1 - 1)(t-a) + m_N Z_1(\tau_1(a)) - m_N[t - g_1(t))] = Z_1(t) - Z_1(a) \geq 0$

and like $t - \tau_1(\tau_2(t)) < t - \tau_1(\tau_2(\tau_1(\tau_2(t))) < t - \tau_1(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t))) < ... < g_1(t))$, we have:

$$(\rho_1 - 1)(t-a) + m_N Z_1(\tau_1(a)) - m_N[t - \tau_1(\tau_2(t))] > 0$$

The equality (45) can be rewritten in the following way:

$$\begin{aligned}
&m_3[Z_2(\tau_2(t))+Z_2(\tau_2^{(2)}(t))+Z_1(\tau_1(\tau_2^{(2)}(t)))]+...+m_{N-2}[Z_2(\tau_2(t))\\
&+Z_1(\tau_1(\tau_2(t)))+Z_1(\tau_1^{(2)}(\tau_2(t)))+Z_2(\tau_2(\tau_1^{(2)}(\tau_2(t))))\\
&+Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2(t))))+...+Z_1(h_2(\tau_2(t)))]+m_{N-1}[Z_2(\tau_2(t))\\
&+Z_2(\tau_2^{(2)}(t))+Z_1(\tau_1(\tau_2^{(2)}(t)))+Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t)))\\
&+Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t))))+Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t))))+...+Z_1(h_3(\tau_2(t)))]\\
&-(\sum_{l_3=6}^{N-2}m_{l_3}+\sum_{l_4=3}^{N-1}m_{l_4})Z_1(\tau_1(a))\\
&=(\rho_2-1)(t-a)+m_2Z_1(\tau_1(a)-m_2(t-\tau_1(\tau_2(t))))
\end{aligned} \tag{48}$$

however

$$[t-\tau_1(\tau_2(t))]<[t-g_1(t))],$$

Thus,

$$\begin{aligned}
&\rho_2Z_1(\tau_1(a))-\rho_2[Z_2(\tau_2(t))+Z_2(\tau_2^{(2)}(t))+Z_1(\tau_1(\tau_2^{(2)}(t)))\\
&+Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t)))+Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t))))+Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t))))\\
&+...+Z_1(g_1(t))]<(1-\rho_2)(t-a).
\end{aligned} \tag{49}$$

In addition the relation (46) implies that

$$\begin{aligned}
&Z_1(t)-Z_1(a)+(1-\rho_1)(t-a)\\
&=m_NZ_1(\tau_1(a))-m_N[Z_2(\tau_2(t))+Z_2(\tau_2^{(2)}(t))+Z_1(\tau_1(\tau_2^{(2)}(t)))\\
&+Z_1(\tau_1^{(2)}(\tau_2^{(2)}(t)))+Z_2(\tau_2(\tau_1^{(2)}(\tau_2^{(2)}(t))))+Z_2(\tau_2^{(2)}(\tau_1^{(2)}(\tau_2^{(2)}(t))))\\
&+...+Z_1(g_1(t))].
\end{aligned} \tag{50}$$

By using the inequality (49), including both terms are negatives, and by taking account that $m_N<\rho_2$, we obtain

$$Z_1(t)-Z_1(a)+(1-\rho_1)(t-a)<(1-\rho_2)(t-a),$$

and since $(1-\rho_2)\leq(1-\rho_1)$, then $Z_1(t)<Z_1(a)$
2nd case: Let $[a,b]\subset(t_{N(m+1)+1},t_{N(m+1)+2})$ such that $Z_2(.)=0$, like $Z_2(.)$ is a positive function, if it is differentiable for all $t\in[a,b]$, then $\dot{Z}_2(t)=0$, or,

$$
\begin{aligned}
Z_2(t) &= \sum_{l_3=2}^{N-2} m_{l_3} Q_{l_3}(t) + \sum_{l_4=3}^{N-1} m_{l_4} Q_{l_4}(t) &&\Rightarrow \dot{Z}_2(t) = \sum_{l_3=2}^{N-2} m_{l_3} \dot{Q}_{l_3}(t) + \sum_{l_4=3}^{N-1} m_{l_4} \dot{Q}_{l_4}(t)\\
& &&\Rightarrow \dot{Q}_{l_3}(t) = \dot{Q}_{l_4}(t),\ \forall\ l_3 = 2...N-2,\\
& &&\forall\ l_4 = 3...N-1,\\
Q_2(t) &= Q_2(0) + \mu_1 T_1(t) - \mu_2 T_2(t) &&\Rightarrow \dot{Q}_2(t) = \mu_1 \dot{T}_1(t) - \mu_2 \dot{T}_2(t) = 0\\
& &&\Rightarrow \mu_1 \dot{T}_1(t) = \mu_2 \dot{T}_2(t)\\
Q_3(t) &= Q_3(0) + \mu_2 T_2(t) - \mu_3 T_3(t) &&\Rightarrow \dot{Q}_3(t) = \mu_2 \dot{T}_2(t) - \mu_3 \dot{T}_3(t) = 0\\
& &&\Rightarrow \mu_2 \dot{T}_2(t) = \mu_3 \dot{T}_3(t)\\
&\vdots\\
&\vdots\\
Q_{N-1}(t) &= Q_{N-1}(0) + \mu_{N-2} T_{N-2}(t)\\
&\quad -\mu_{N-3} T_{N-3}(t) &&\Rightarrow \dot{Q}_{N-1}(t) = \mu_{N-2} \dot{T}_{N-2}(t)\\
& &&-\mu_{N-3} \dot{T}_{N-3}(t) = 0\\
& &&\Rightarrow \mu_{N-2} \dot{T}_{N-2}(t) = \mu_{N-3} \dot{T}_{N-3}(t)
\end{aligned}
$$

this implies

$$\mu_1 \dot{T}_1(t) = \mu_2 \dot{T}_2(t) = \mu_3 \dot{T}_3(t) = ... = \mu_{N-1} \dot{T}_{N-1}(t).$$

Thus we have on one hand:

$$
\begin{aligned}
&Z_1(t) = m_1[Q_1(0) + t - \mu_1 T_1(t)] + \sum_{l_2=5}^{N-3} m_{l_2}[Q_{l_2}(0) + \mu_{l_2-1} T_{l_2-1}(t) - \mu_{l_2} T_{l_2}(t)]\\
&+ \sum_{l_1=4}^{N} m_{l_1}[Q_{l_1}(0) + \mu_{l_1-1} T_{l_1-1}(t) - \mu_{l_1} T_{l_1}(t)]\\
&\Rightarrow \dot{Z}_1(t) = m_1[1 - \mu_1 \dot{T}_1(t)] + \sum_{l_2=5}^{N-3} m_{l_2}[\mu_{l_2-1} \dot{T}_{l_2-1}(t) - \mu_{l_2} \dot{T}_{l_2}(t)]\\
&+ \sum_{l_1=4}^{N} m_{l_1}[\mu_{l_1-1} \dot{T}_{l_1-1}(t) - \mu_{l_1} \dot{T}_{l_1}(t)] = 0\\
&= m_1 + m_4 \mu_3 \dot{T}_3(t) + m_5 \mu_4 \dot{T}_4(t) + ... + m_N \mu_{N-1} \dot{T}_{N-1}(t) - [\dot{T}_1(t) - \dot{T}_4(t)\\
&-\dot{T}_5(t) - ... - \dot{T}_N(t)]\\
&= m_1 + \sum_{l_1=4}^{N} m_{l_1} \mu_{l_1-1} \dot{T}_{l_1-1}(t) + \sum_{l_2=5}^{N-3} m_{l_2} \mu_{l_2-1} \dot{T}_{l_2-1}(t) - \dot{B}_1(t).
\end{aligned}
$$

And in addition

$$\dot{Z}_2(t) = \sum_{l_3=2}^{N-2} m_{l_3}[\mu_{l_3-1} \dot{T}_{l_3-1}(t) - \mu_{l_3} \dot{T}_{l_3}(t)] + \sum_{l_4=3}^{N-1} m_{l_4}[\mu_{l_4-1} \dot{T}_{l_4-1}(t) - \mu_{l_4} \dot{T}_{l_4}(t)] - \dot{B}_2(t) = 0.$$

Thus

$$\rho_2 \mu_1 \dot{T}_1(t) = \dot{B}_2(t) < 1.$$

Since $m_N < \rho_2$ we obtain

$$Z_1(t) - Z_1(a) = \int_a^t \dot{Z}_1(u)du \leq 0.$$

Let us recapitulate above, for all $i = 0, ..., M-1$

$$Z_1(t) \leq Z_1(a_i) \text{ if } t \in [a_i, a_{i+1}]$$

thus, for all $t \in [a_0, a_M] = [t_{Nm+(N+1)}, t_{N(m+1)+2}], Z_1(t) \leq Z_1(a_0) = Z_1(t_{Nm+(N+1)})$ and by the second inequality

$$Z_1(t) < m_N[Z_2(g_2(t_{Nm+N})) + Z_1(g_1(t_{Nm+N})))].$$

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