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SHARP WEIGHTED INEQUALITIES FOR VECTOR-VALUED MULTILINEAR COMMUTATORS OF MARCINKIEWICZ OPERATOR

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ABSTRACT. In this paper, we prove the sharp inequality for the vector-valued multilinear commutators related to the Marcinkiewicz operator. By using the sharp inequality, we obtain the weighted L^p-norm inequality for the vector-valued multilinear commutators.

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1. INTRODUCTION

Let T be the Calderón-Zygmund singular integral operator, we know that the commutator $[b,T](f) = T(bf) - bT(f)$ (where $b \in BMO(R^n)$) is bounded on $L^p(R^n)$ for $1 < p < \infty$ (see [3]). In [9], the sharp estimates for some multilinear commutators of the Calderón-Zygmund singular integral operators are obtained. The main purpose of this paper is to prove a sharp inequality for some vector-valued multilinear commutators related to the Marcinkiewicz operator. By using the sharp inequality, we obtain the weighted L^p-norm inequality for the vector-valued multilinear commutators.

2. NOTATIONS AND RESULTS

First let us introduce some notations(see [4][9][10]). In this paper, Q will denote a cube of R^n with sides parallel to the axes, and for a cue Q let $f_Q = |Q|^{-1}\int_Q f(z)dz$ and the sharp function of f is defined by

$$f^{\#}(x) = \sup_{Q \ni x} \frac{1}{|Q|}\int_Q |f(y) - f_Q|dy.$$

It is well-known that(see [4])

$$f^{\#}(x) = \sup_{Q \ni x} \inf_{c \in C} \frac{1}{|Q|} \int_Q |f(y) - c| dy.$$

We say that b belongs to $BMO(R^n)$ if $b^{\#}$ belongs to $L^{\infty}(R^n)$ and define $||b||_{BMO} = ||b^{\#}||_{L^{\infty}}$. If $\overrightarrow{b} = (b_1, \cdots, b_m)$, $b_j \in BMO$ for $(j = 1, \cdots, m)$, then

$$||\vec{b}||_{BMO} = \prod_{j=1}^{m} ||b_j||_{BMO}.$$

Let M be the Hardy-Littlewood maximal operator, that is that

$$M(f)(x) = \sup_{x \in Q} |Q|^{-1} \int_Q |f(y)| dy;$$

we write that $M_p(f) = (M(|f|^p))^{1/p}$ for $0 < p < \infty$.

Given a positive integer m and $1 \leq j \leq m$, we denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \cdots, \sigma(j)\}$ of $\{1, \cdots, m\}$ of j different elements. For $\sigma \in C_j^m$, set $\sigma^c = \{1, \cdots, m\} \setminus \sigma$. For $\vec{b} = (b_1, \cdots, b_m)$ and $\sigma = \{\sigma(1), \cdots, \sigma(j)\} \in C_j^m$, set $\vec{b}_\sigma = (b_{\sigma(1)}, \cdots, b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)}$ and $||\vec{b}_\sigma||_{BMO} = ||b_{\sigma(1)}||_{BMO} \cdots ||b_{\sigma(j)}||_{BMO}$.

We denote the Muckenhoupt weights by A_p, let $\Omega \in A_p$ and $1 \leq p < \infty$, ω satisfy the inverse Hölder inequality, there exists a constant C and $1 < q < \infty$, for any cube Q, we get(see [10])

$$\left(\frac{1}{|Q|} \int_Q \omega(x)^q dx \right)^{1/q} \leq \frac{C}{|Q|} \int_Q \omega(x) dx.$$

In this paper, we will study some vector-valued multilinear commutators as following.

Definition. *Let $0 < \gamma \leq 1$ and Ω be homogeneous of degree zero on R^n such that $\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0$. Assume that $\Omega \in Lip_\gamma(S^{n-1})$, that is there exists a constant $M > 0$ such that for any $x, y \in S^{n-1}$, $|\Omega(x) - \Omega(y)| \leq M|x - y|^\gamma$. Set $b_j (j = 1, \cdots, m)$ as a fixed locally integrable function of R^n, then when $1 < r < \infty$, The vector-valued Marcinkiewicz multilinear commutators is defined by*

$$|\mu_\Omega^{\vec{b}}(f)(x)| = (\sum_{i=1}^{\infty} (\mu_\Omega^{\vec{b}}(f_i)(x))^r)^{1/r},$$

where

$$\mu_\Omega^{\vec{b}}(f)(x) = \left(\int_0^\infty |F_t^{\vec{b}}(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2}$$

and

$$F_t^{\vec{b}}(f)(x)=\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}\left[\prod_{j=1}^{m}(b_j(x)-b_j(y))\right]f(y)dy.$$

Set

$$F_t(f)(x)=\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}f(y)dy,$$

we also define that

$$\mu_\Omega(f)(x)=\left(\int_0^\infty |F_t(f)(x)|^2\frac{dt}{t^3}\right)^{1/2},$$

which is the Marcinkiewicz operator(see [11]).

Let H be the space $H=\left\{h:||h||=\left(\int_0^\infty |h(t)|^2dt/t^3\right)^{1/2}<\infty\right\}$. Then, it is clear that

$$\mu_\Omega(f)(x)=||F_t(f)(x)|| \text{ and } \mu_\Omega^{\tilde{b}}(f)(x)=||F_t^{\tilde{b}}(f)(x)||.$$

Note that when $b_1=\cdots=b_m$, $|\mu_\Omega^{\vec{b}}(f)(x)|$ is just the m order vector-valued Marcinkiewicz operator multilinear commutators. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [1][4-8][10]). Our main purpose is to establish the sharp inequality for the vector-valued Marcinkiewicz operator multilinear commutators.

Now we state our main results as following.

3. Main Theorem and proof

First, we will establish the following theorem.

Theorem 1. *Let $1<r<\infty$, $b_j\in BMO(R^n)$ for $j=1,\cdots,m$. Then for any $1<s<\infty$, there exists a constant $C>0$ such that for any $f\in C_0^\infty(R^n)$ and any $\widetilde{x}\in R^n$,*

$$(|\mu_\Omega^{\vec{b}}(f)|_r)^{\#}(\widetilde{x})\le C\left(||\vec{b}||_{BMO}M_s(|f|_r)(\widetilde{x})+\sum_{j=1}^{m}\sum_{\sigma\in C_j^m}||\vec{b}_\sigma||_{BMO}M_s(|\mu_\Omega^{\vec{b}_{\sigma^c}}(f)|_r)(\widetilde{x})\right).$$

Theorem 2. *Let $1<r<\infty$, $b_j\in BMO(R^n)$ for $j=1,\cdots,m$. Then $|\mu_\Omega^{\vec{b}}|_r$ is bounded on $L^p(R^n)$ for $1<p<\infty$.*

To proof the theorem,we need the following lemmas.

Lemma 1. *(see [11]) Let $w \in A_p$ and $1 < r < \infty$, $1 < p < \infty$. When $\Omega \in Lip_r(S^{n-1})$ for $0 < \gamma \leq 1$, then $|\mu_\Omega|_r$ is bounded on $L^p(w)$.*

Lemma 2. *Let $1 < r < \infty$, $b_j \in BMO$ for $j = 1, \cdots, k$ and $k \in N$. Then, we have*

$$\frac{1}{|Q|}\int_Q \prod_{j=1}^k |b_j(y) - (b_j)_Q| dy \leq C \prod_{j=1}^k ||b_j||_{BMO}$$

and

$$\left(\frac{1}{|Q|}\int_Q \prod_{j=1}^k |b_j(y) - (b_j)_Q|^r dy\right)^{1/r} \leq C \prod_{j=1}^k ||b_j||_{BMO}.$$

Proof. For $\sigma \in C_k^m$, where $k \leq m$ and $m \in N$, we have

$$\frac{1}{|Q|}\int_Q |(b(y) - (b_j)_Q)_\sigma| dy \leq C||b_\sigma||_{BMO}$$

and

$$\left(\frac{1}{|Q|}\int_Q |(b(y) - (b_j)_Q)_\sigma|^r dy\right)^{1/r} \leq C||b_\sigma||_{BMO}.$$

We just need to choose $p_j > 1$ and $q_j > 1$, where $1 \leq j \leq k$, such that $1/p_1 + \cdots + 1/p_k = 1$ and $1/q_1 + \cdots + 1/q_k = 1/r$. After that,using the Hölder's inequality with exponent $1/p_1 + \cdots + 1/p_k = 1$ and $1/q_1 + \cdots + 1/q_k = 1/r$ respectively, we may get the conclusions.

Lemma 3.*(see [11]) Let $0 < \gamma \leq 1$ and Ω be homogeneous of degree zero on R^n such that $\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0$. Assume that $\Omega \in Lip_\gamma(S^{n-1})$, if $Q = Q(x_0, d), y \in (2Q)^c$, then*

$$\left|\frac{\Omega(x-y)}{|x-y|^{n-1}} - \frac{\Omega(x_0-y)}{|x_0-y|^{n-1}}\right| \leq C\left(\frac{|x-x_0|}{|x_0-y|^n} + \frac{|x-x_0|^\gamma}{|x_0-y|^{n-1+\gamma}}\right).$$

Proof of Theorem 1. It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\left(\frac{1}{|Q|}\int_Q ||\mu_\Omega^{\vec{b}}(f)(x)|_r - C_0| dx\right) \leq C\left(M_s(|f|_r)(\widetilde{x}) + \sum_{j=1}^m \sum_{\sigma \in C_j^m} M_s(|\mu_\Omega^{\vec{b}}(f)|_r)(\widetilde{x})\right).$$

Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. We write, $f = g + h = g_i + h_i$, where $g_i = f_i\chi_{2Q}$ and $h_i = f_i\chi_{(2Q)^c}$. If let $\tilde{b} = (b_1, ..., b_m)$, where $(b_j)_Q = |Q|^{-1}\int_Q b_j(y)dy$, for $1 \le j \le m$. We have

$$
\begin{aligned}
F_t^{\tilde{b}}(f_i)(x) &= \int_{|x-y|\le t}\left[\prod_{j=1}^m (b_j(x) - b_j(y))\right] f_i(y)\frac{\Omega(x-y)}{|x-y|^{n-1}}dy \\
&= \int_{|x-y|\le t}\left[\prod_{j=1}^m ((b_j(x) - (b_j)_{2Q}) - (b_j(y) - (b_j)_{2Q}))\right] f_i(y)\frac{\Omega(x-y)}{|x-y|^{n-1}}dy \\
&= \sum_{j=0}^m \sum_{\sigma\in C_j^m} (-1)^{m-j}(b(x) - (b)_{2Q})_\sigma \int_{|x-y|\le t}(b(y) - (b)_{2Q})_{\sigma^c} f_i(y)\frac{\Omega(x-y)}{|x-y|^{n-1}}dy \\
&= (b_1(x) - (b_1)_{2Q})\cdots(b_m(x) - (b_m)_{2Q})F_t(f_i)(x) \\
&\quad +(-1)^m F_t((b_1 - (b_1)_{2Q})\cdots(b_m - (b_m)_{2Q})f_i)(x) \\
&\quad +\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x) - (b)_{2Q})_\sigma F_t^{\tilde{b}_{\sigma^c}}(f_i)(x),
\end{aligned}
$$

Then by Minkowski's inequality, we get

$$\begin{aligned}
& \frac{1}{|Q|}\int_Q ||\mu_\Omega^{\vec{b}}(f)(x)|_r - |\mu_\Omega((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q}))h)(x_0)|_r|dx \\
\leq & \frac{1}{|Q|}\int_Q |||F_t^{\vec{b}}(f)(x)||_r - ||F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q}))h)(x_0)||_r|dx \\
\leq & \frac{1}{|Q|}\int_Q \left(\sum_{i=1}^\infty ||F_t^{\vec{b}}(f_i)(x) - F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})h_i)(x_0)||^r\right)^{1/r} dx \\
\leq & \frac{1}{|Q|}\int_Q \left(\sum_{i=1}^\infty ||(b_1(x)-(b_1)_{2Q})\cdots(b_m(x)-(b_m)_{2Q})F_t(f_i)(x)||^r\right)^{1/r} dx \\
& +\frac{1}{|Q|}\int_Q \left(\sum_{i=1}^\infty \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m} ||(b(x)-(b)_{2Q})_\sigma F_t^{\tilde{b}_{\sigma^c}}(f_i)(x)||^r\right)^{1/r} dx \\
& +\frac{1}{|Q|}\int_Q \left(\sum_{i=1}^\infty ||F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})g_i)(x)||^r\right)^{1/r} \\
& +\frac{1}{|Q|}\int_Q \left(\sum_{i=1}^\infty ||F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})h_i)(x)\right. \\
& \left. -F_t((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})h_i)(x_0)||^r\right)^{1/r} dx \\
= & I_1+I_2+I_3+I_4.
\end{aligned}$$

For I_1, by Hölder's inequality with exponent $1/s'+1/s=1$ and lemma 2, we get

$$\begin{aligned}
I_1 \leq & C\frac{1}{|Q|}\int_Q |\prod_{j=1}^m (b_j(x)-(b_j)_{2Q})||\mu_\Omega(f)(x)|_r dx \\
\leq & C\left(\frac{1}{|2Q|}\int_{2Q} |\prod_{j=1}^m (b_j(x)-(b_j)_{2Q})|^{s'}dx\right)^{1/s'} \left(\frac{1}{|Q|}\int_Q |\mu_\Omega(f)(x)|_r^s dx\right)^{1/s} \\
\leq & C||\vec{b}||_{BMO} M_s(|\mu_\Omega(f)|_r)(\tilde{x}).
\end{aligned}$$

For I_2, by Hölder's inequality with exponent $1/s'+1/s=1$, we have

$$\begin{aligned}
I_2 &= \frac{1}{|Q|}\int_Q \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m} ||(b(x)-(b)_{2Q})_\sigma F_t^{\vec{b}_{\sigma^c}}(f)(x)||_r dx \\
&\leq \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m} \frac{1}{|Q|}\int_Q |(b(x)-(b)_{2Q})_\sigma||\mu_\Omega^{\vec{b}_{\sigma^c}}(f)(x)|_r dx \\
&\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\left(\frac{1}{|2Q|}\int_{2Q}|(b(x)-(b)_{2Q})_\sigma|^{s'}dx\right)^{1/s'}\left(\frac{1}{|Q|}\int_Q|\mu_\Omega^{\vec{b}_{\sigma^c}}(f)(x)|_r^s dx\right)^{1/s} \\
&\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||\vec{b}_\sigma||_{BMO}M_s(|\mu_\Omega^{\vec{b}_{\sigma^c}}(f)|_r)(\tilde{x}).
\end{aligned}$$

For I_3, we choose some p, such that $1<p<s$, by the boundness of $|\mu_\Omega|_r$ on $L^p(R^n)$(see lemma 1) and Hölder's inequality, we gain

$$\begin{aligned}
I_3 &= \frac{1}{|Q|}\int_Q ||F_t(\prod_{j=1}^m (b_j(y)-(b_j)_{2Q})g)(x)||_r dx \\
&\leq \left(\frac{1}{|Q|}\int_{R^n}|\mu_\Omega(\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})f\chi_{2Q})(x)|_r^p dx\right)^{1/p} \\
&\leq C\left(\frac{1}{|Q|}\int_{R^n}|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})|^p|f\chi_{2Q}|_r^p dx\right)^{1/p} \\
&\leq C\left(\frac{1}{|2Q|}\int_{2Q}|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})|^{sp/(s-p)}dx\right)^{(s-p)/sp}\left(\frac{1}{|2Q|}\int_{2Q}|f(x)|_r^s dx\right)^{1/s} \\
&\leq C||\vec{b}||_{BMO}M_s(|f|_r)(\tilde{x}).
\end{aligned}$$

For I_4, we have

$$\begin{aligned}
&||F_t(\prod_{j=1}^{m}(b_j-(b_j)_{2Q})h)(x)-F_t(\prod_{j=1}^{m}(b_j-(b_j)_{2Q})h)(x_0)||_r\\
= &\left(\int_0^\infty |\int_{|x-y|\le t}\frac{\Omega(x-y)|h(y)|_r}{|x-y|^{n-1}}\left[\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})\right]dy\right.\\
&\left.-\int_{|x_0-y|\le t}\frac{\Omega(x_0-y)|h(y)|_r}{|x_0-y|^{n-1}}\left[\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})\right]dy|^2\frac{dt}{t^3}\right)^{1/2}\\
\le &\left(\int_0^\infty\left[\int_{|x_0-y|\le t,|x_0-y|>t}\frac{|\Omega(x-y)||h(y)|_r}{|x-y|^{n-1}}\left|\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})\right|dy\right]^2\frac{dt}{t^3}\right)^{1/2}\\
&+\left(\int_0^\infty\left[\int_{|x-y|>t,|x_0-y|\le t}\frac{|\Omega(x_0-y)||h(y)|_r}{|x_0-y|^{n-1}}\left|\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})\right|dy\right]^2\frac{dt}{t^3}\right)^{1/2}\\
&+\left(\int_0^\infty\left[\int_{|x-y|\le t,|x_0-y|\le t}\left|\frac{|\Omega(x-y)|}{|x-y|^{n-1}}-\frac{|\Omega(x_0-y)|}{|x_0-y|^{n-1}}\right|\right.\right.\\
&\left.\left.\times\left|\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})\right||h(y)|_r dy\right]^2\frac{dt}{t^3}\right)^{1/2}\\
\equiv &\ J_1+J_2+J_3.
\end{aligned}$$

For J_1, we get

$$\begin{aligned}
J_1 &\leq C\int_{(2Q)^c}\left|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\right|\frac{|f(y)|_r}{|x-y|^{n-1}}\left(\int_{|x-y|\leq t<|x_0-y|}\frac{dt}{t^3}\right)^{1/2}dy\\
&\leq C\int_{(2Q)^c}\left|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\right|\frac{|f(y)|_r}{|x-y|^{n-1}}\left|\frac{1}{|x-y|^2}-\frac{1}{|x_0-y|^2}\right|^{1/2}dy\\
&\leq C\int_{(2Q)^c}\left|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\right|\frac{|f(y)|_r}{|x-y|^{n-1}}\frac{|x_0-x|^{1/2}}{|x-y|^{3/2}}dy\\
&\leq C\sum_{k=1}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}\left|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\right|\frac{|Q|^{1/(2n)}|f(y)|_r}{|x_0-y|^{n+1/2}}dy\\
&\leq C\sum_{k=1}^{\infty}2^{-k/2}|2^{k+1}Q|^{-1}\int_{2^{k+1}Q}\left|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\right||f(y)|_r dy\\
&\leq C\sum_{k=1}^{\infty}2^{-k/2}\left(|2^{k+1}Q|^{-1}\int_{2^{k+1}Q}\left|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\right|^{s'}dy\right)^{1/s'}\\
&\quad\times\left(|2^{k+1}Q|^{-1}\int_{2^{k+1}Q}|f(y)|_r^s dy\right)^{1/s}\\
&\leq C\sum_{k=1}^{\infty}2^{-k/2}\prod_{j=1}^m||b_j||_{BMO}M_s(|f|_r)(\tilde{x})\\
&\leq C||\vec{b}||_{BMO}M_s(|f|_r)(\tilde{x}).
\end{aligned}$$

Similarly, we have $J_2\leq C||\vec{b}||_{BMO}M_s(|f|_r)(\tilde{x})$.

We now estimate J_3, by the Lemma 3, we gain

$$\begin{aligned} J_3 &\le C\int_{(2Q)^c}\left|\prod_{j=1}^m (b_j(y)-(b_j)_{2Q})\right| \frac{|f(y)|_r|x-x_0|}{|x_0-y|^n}\left(\int_{|x_0-y|\le t,|x-y|\le t}\frac{dt}{t^3}\right)^{1/2} dy \\ &+C\int_{(2Q)^c}\left|\prod_{j=1}^m (b_j(y)-(b_j)_{2Q})\right| \frac{|f(y)|_r|x-x_0|^\gamma}{|x_0-y|^{n-1+\gamma}}\left(\int_{|x_0-y|\le t,|x-y|\le t}\frac{dt}{t^3}\right)^{1/2} dy \\ &\le C\sum_{k=1}^\infty\int_{2^{k+1}Q\setminus 2^kQ}\left|\prod_{j=1}^m (b_j(y)-(b_j)_{2Q})\right|\left(\frac{|Q|^{1/n}}{|x_0-y|^{n+1}}+\frac{|Q|^{\gamma/n}}{|x_0-y|^{n+\gamma}}\right)|f(y)|_r dy \\ &\le C\sum_{k=1}^\infty (2^{-k}+2^{-k\gamma})|2^{k+1}Q|^{-1}\int_{2^{k+1}Q}\left|\prod_{j=1}^m (b_j(y)-(b_j)_{2Q})\right| |f(y)|_r dy \\ &\le C\sum_{k=1}^\infty (2^{-k}+2^{-k\gamma})\prod_{j=1}^m ||b_j||_{BMO}M_s(|f|_r)(\tilde{x}) \\ &\le C||\vec{b}||_{BMO}M_s(|f|_r)(\tilde{x}). \end{aligned}$$

This completes the total proof of Theorem 1.

Proof of Theorem 2. We first consider the case $m=1$, for $1<p<\infty$, Choose s such that $1<s<p$, by using Theorem 1 and Lemma 1, we may get

$$\begin{aligned} |||\mu_\Omega^{b_1}(f)|_r||_{L^p} &\le \|M(|\mu_\Omega^{b_1}(f)|_r\|_{L^p} \le C\|(|\mu_\Omega^{b_1}(f)|_r)^{\#}\|_{L^p} \\ &\le C\|M_s(|\mu_\Omega^{b_1}(f)|_r\|_{L^p} + C\|M_s(|f|_r))\|_{L^p} \\ &\le C\||\mu_\Omega^{b_1}(f)|_r\|_{L^p} + C\||f|_r\|_{L^p} \\ &\le C\||f|_r\|_{L^p}. \end{aligned}$$

When $m\ge 2$, we may obtain the conclusion by induction. This finishes the proof.

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