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SIMULTANEOUS APPROXIMATION BY A CLASS OF LINEAR POSITIVE OPERATORS

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Abstract.Some direct theorems for the linear combinations of a new class of positive linear operators have been obtained for both, pointwise and uniform simultaneous approximations. A number of well known positive linear operators such as Gamma-Operators of Muller, Post-Widder and modified Post-Widder Operators are special cases of this class of operators.

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1. Introduction

Let G be a non negative function measurable on positive real line $IR^+ \equiv (0, \infty)$,which is continuousat the point $'1'$, and satisfies the following properties:

(i) for each $\delta > 0$, $\left\| \chi^c_{\delta,1} G \right\|_\infty < G(1)$, and

(ii) there exists $\theta_1, \theta_2 > 0$ such that $(u^{\theta_1} + u^{-\theta_2})G(u)$

is essentially bounded. Such a function will be called an "admissible" kernel function. The set of admissible kernel functions will be denoted by $T(IR^+)$. Throughout the paper $\chi_{\delta,x}(\chi^c_{\delta,x})$ denotes the characteristic function of $(x-\delta, x+\delta)\{IR^+ - (x-\delta, x+\delta)\}$.

Let $G \in T(IR^+), \alpha \in IR$. Then for $\lambda, x \in IR^+$ and a non negative function f measurable on IR^+, we define

(1) $$T_\lambda(f;x) = \frac{x^{\alpha-1}}{a(\lambda)} \int_0^\infty u^{-\alpha} f(u) G^\lambda(xu^{-1}) du,$$

where $a(\lambda) = \int_0^\infty u^{\alpha-2} G^\lambda(u) du$,

whenever the above integral exists.

The equation (1) defines a class of linear positive approximation methods, which contains as particular cases, a number of well known linear positive operators; e.g. Post Widder and modified Post Widder Operators [5] and the Gamma-Operators of Muller [6] etc., as shown in [3].

In the present paper we study the following problems:

(i) Is it possible to approximate the derivatives of f by the derivatives of $T_\lambda(f)$?

(ii) Can we use certain linear combinations of T_λ to obtain a better order of approximation?

We introduce notations and definitions used in this paper.

Let $\Omega(>1)$ be a continuous function defined on IR^+. We call Ω a bounding function [8] for G if for each compact $K \subseteq IR^+$ there exist positive numbers λ_K and M_K such that

$$T_{\lambda_K}(\Omega;x) < M_K,\ x \in K.$$

It is clear that if $G \in T(IR^+)$, then $\Omega(u) = u^p + u^{-q}$ for $p, q > 0$ is a bounding function for G. The notion of a bounding function enables us to obtain results in a uniform set-up, which, at the same time , are applicable for a general $G \in T(IR^+)$.

For a bounding function Ω, we define the set

$D_\Omega = \{f : f$ is locally integrable on IR^+ and is such that $\limsup_{u\to 0} \frac{f(u)}{\Omega(u)}$ and $\limsup_{u\to\infty} \frac{f(u)}{\Omega(u)}$ exist$\}$

$D_\Omega^{(k)} = \{f : f \in D_\Omega$ and f is k-times continuously differentiable on IR^+ and $f^{(i)} \in D_\Omega, i = 1, 2, ...k\}$

$C_b^{(m)}(IR^+) = \{f : f$ is $m-$times continuously differentiable and is such that $f^{(k)}, k = 0, 1, 2, ...m$, are bounded on $IR^+\}$

$T_\infty(IR^+) = \{G \in T(IR^+) : G$ is continuously differentiable at u=1 and $G''(1) \neq 0\}$

2.Simultaneous Approximation for Continuous Derivatives

We consider first the "elementary" case of simultaneous approximation by the operatorsT_λ wherein the derivatives of f are assumed to be continuous. We have termed this case elementary, for it is possible here to deduce the results on the simultaneous approximation :$(T_\lambda f)^{(k)} \to f^{(k)} (k \in IN)$ from the corresponding results on the ordinary approximation : $T_\lambda f \to f$. Indeed, the operators $T_\lambda f$ become differentiable either by assuming the differentiability of G or that of f. The situation of the present section corresponds to the latter case.

Theorem 1 *-If $f \in D_\Omega^{(k)}$, then $T_\lambda^{(k)}(f;x)$ for $x \in [a,b]$ exist for all sufficiently large λ and*

(2) $\quad \lim_{\lambda\to\infty} T_\lambda^{(k)}(f;x) = f^{(k)}(x)$, *uniformly for $x \in [a,b]$.*

Proof. - We have

$$T_\lambda(f;x) = \frac{1}{a(\lambda)} \int_0^\infty u^{\alpha-2} G^\lambda(u) f(xu^{-1}) du.$$

A formal k-times differentiation within the integral sign leads to

$$T_\lambda^{(k)}(f;x) = \frac{x^{\alpha-k-1}}{a(\lambda)}\int_0^\infty u^{-(\alpha-k)}G^\lambda(xu^{-1})f^{(k)}(u)du.$$

It follows that $T_\lambda^{(k)}(f;x), x \in [a,b]$, exist for all λ sufficiently large.

Let T_λ^* denote the operator obtained from (1) after replacing α by $\alpha - k$. Let the corresponding $a(\lambda)$ be denoted by $a^*(\lambda)$. Then we have

(3) $\quad T_\lambda^{(k)}(f;x) = \frac{a^*(\lambda)}{a(\lambda)}T_\lambda^*(f^{(k)};x).$

We also note that

(4) $\quad \frac{a^*(\lambda)}{a(\lambda)} = T_\lambda(u^k;1)$, as $\lambda \to \infty$.

Applying the known approximation $T_\lambda f \to f$ to (3), we find that

$$T_\lambda^{(k)}(f;x) = \frac{a^*(\lambda)}{a(\lambda)}T_\lambda^*(f^{(k)};x) \to f^{(k)}(x) \text{ as } \lambda \to \infty.$$

This completes the proof of the theorem.

Theorem 2 - *Let $G \in T(IR^+), G'''(1)$ exist and $G''(1)$ be non-zero and $f \in D_\Omega^{(k)}$. Then, at each $x \in IR^+$ where $f^{(k+2)}$ exists,*

(5) $$T_\lambda^{(k)}(f;x) - f^{(k)}(x) = \frac{1}{2\lambda[G''(1)]^2}[f^{(k)}(x)kG(1)\{(2\alpha-k-5)G''(1) - G'''(1)\} + xf^{(k+1)}(x)G(1)\{2(\alpha-k-3)G''(1) - G'''(1)\} - x^2f^{(k+2)}(x)G(1)G''(1)] + o(\tfrac{1}{\lambda}), \lambda \to \infty.$$

Further, if $f^{(k+2)}$ exists and is continuous on $< a,b >$, the open interval containing $[a,b]$, then (5) holds uniformly in $x \in [a,b]$.

Proof. Using Voronovskaya formula [3] [8] [5] [9] [10] for T_λ^* and (3),the result follows.

In the similar manner, one can prove the following results:

Theorem 3 - *Let $G \in T(IR^+)$ and $G''(1)$ be non-zero. If f is such that $f^{(k)}$ exists on IR^+ and is continuous on IR^+, then*

(6) $$\left|T_\lambda^{(k)}(f;x) - f^{(k)}(x)\right| \leq .\omega_{f^{(k)}}(\lambda^{-\frac{1}{2}})[1 + \min(x^2\{-\tfrac{G(1)}{G''(1)} + o(1)\}, x\{-\tfrac{G(1)}{G''(1)} + o(1)\}^{\frac{1}{2}})] + o(\tfrac{1}{\lambda}), (\lambda \to \infty, x \in IR^+),$$

where $\omega_{f^{(k)}}$ is the modulus of continuity of $f^{(k)}$,[11]

Theorem 4 -*With the same assumption on G as in* ***Theorem2,*** *letf be such thatf$^{(k+1)}$ exists on IR^+. Then, for $x \in IR^+$*

(7) $$\left|T_\lambda^{(k)}(f;x) - f^{(k)}(x)\right| \leq \frac{k\left|f^{(k)}(x)\right|}{2\lambda[G''(1)]^2}\{G(1)\left|(2\alpha-k-5)G''(1) - G'''(1)\right|\} + \frac{x\left|f^{(k+1)}(x)\right|}{2\lambda[G''(1)]^2}\{G(1)\left|2(\alpha-k-3)G''(1) - G'''(1)\right|\}$$

$$+o(\frac{1}{\lambda}) + \omega_{f^{(k+1)}}(\lambda^{-\frac{1}{2}})[\frac{x}{\lambda^{\frac{1}{2}}}\{(-\frac{G(1)}{G'(1)})^{\frac{1}{2}} + o(1)\} + \frac{x^2}{2\lambda^{\frac{1}{2}}}\{-\frac{G(1)}{G''(1)} + o(1)\}], \quad (\lambda \to \infty, x \in IR^+).$$

3.Pointwise Simultaneous Approximation

In the present section we consider the "non-elementary" case of simultaneous approximation wherein only G is assumed to be sufficiently smooth. Then, assuming only that $f^{(k)}(x)$ exists at some point x, we solve the problem of point-wise approximation. Before proving this result , we establish:

Lemma 5 *1- Let $G \in C_b^{(m)}(IR^+) \cap T(IR^+)$ and $\lambda > m \in IN$ (set of natural numbers). Then*

$$(8) \quad \frac{\partial^m}{\partial x^m}\{x^{\alpha-1}G^{\lambda}(xu^{-1})\} = x^{\alpha-1}G^{\lambda-m}(xu^{-1})\sum_{k=0}^{m}\sum_{\upsilon=0}^{[\frac{m-k}{2}]}\lambda^{\upsilon+k}\{G'(xu^{-1})\}^k g_{\upsilon,k,m}(x,u)$$

where $[x]$ denotes the integral part of $x \in IR^+$ and the function $g_{\upsilon,k,m}(x,u)$ are certain linear combinations of products of the powers of u^{-1}, x^{-1} and $G^{(k)}(xu^{-1}), k = 0, 1, 2, ...m$ and are independent of λ.

Proof. - We proceed by induction on m. We note that

$$(9) \quad \frac{\partial}{\partial x}\{x^{\alpha-1}G^{\lambda}(xu^{-1})\} = x^{\alpha-1}G^{\lambda-1}(xu^{-1})[\frac{\alpha-1}{x}G(xu^{-1}) + \frac{\lambda}{u}G'(xu^{-1})].$$

Putting $g_{0,0,1}(x,u) = \frac{\alpha-1}{x}G(xu^{-1})$ and $g_{0,1,1}(x,u) = \frac{1}{u}$, we observe that (9) is of the form (8). Hence the result is true for $m = 1$.

Next, let us assume that the lemma holds for a certain m. Let $G \in C_b^{(m+1)}(IR^+) \cap T(IR^+)$. Then $G \in C_b^{(m)}(IR^+) \cap T(IR^+)$ and therefore by the induction hypothesis,

$$(10) \quad \frac{\partial^{m+1}}{\partial x^{m+1}}\{x^{\alpha-1}G^{\lambda}(xu^{-1})\} = x^{\alpha-1}G^{\lambda-m-1}(xu^{-1})\sum_{k=0}^{m+1}\sum_{\upsilon=0}^{[\frac{m+1-k}{2}]}\lambda^{\upsilon+k}\{G'(xu^{-1})\}^k \cdot g_{\upsilon,k,m+1}(x,u),$$

wherewith $g_{\upsilon,k,m}(x,u) \equiv 0$ for $k > m$ or $k < 0, \upsilon < 0$ or $\upsilon > [\frac{m-k}{2}]$, we have put

$$g_{\upsilon,k,m+1}(x,u) = \frac{\alpha-1}{x}g_{\upsilon,k,m}(x,u)G(xu^{-1}) - \frac{\lambda}{u}G'(xu^{-1})g_{\upsilon,k,m}(x,u) + \frac{\partial}{\partial x}g_{\upsilon,k,m}(x,u) + \frac{1}{u}g_{\upsilon,k-1,m}(x,u) + \frac{k+1}{u}G''(xu^{-1})g_{\upsilon-1,k+1,m}(x,u),$$

for $k = 0, 1, 2, ..., m+1$ and $\upsilon = 0, 1, 2, ..., [\frac{m+1-k}{2}]$

It is clear that $g_{\upsilon,k,m+1}(x,u)$ satisfies the other required properties and hence the result is true for m+1. Hence it follows that (8) holds for all $m = 1, 2, ...$ This completes the proof.

Remark 1 *: It can be seen that for $G \in C_b^{(m)}(IR^+) \cap T(IR^+)$ and$f \in D_\Omega$, where Ω is some bounding function for $G, T_\lambda^{(k)}(f;x)$ exist where $x \in [a,b], 0 < a < b < \infty$ and $k = 1, 2, 3, ...m$.*

Theorem 6 *5: Let $m \in IN, G \in C_b^{(m)}(IR^+)$ and $G"(1)$ exist and be non-zero. If Ω is a bounding function for G and$f \in D_\Omega$, then*

(11) $\lim_{\lambda\to\infty} T_\lambda^{(m)}(f;x) = f^{(m)}(x)$,

whenever $x \in IR^+$ is such that$f^{(m)}(x)$ exists. Moreover if$f^{(m)}$ exists and is continuous on $< a, b >$, (11) holds uniformly in $x \in [a,b]$.

Proof. If $f^{(m)}(x)$ exists at some $x \in IR^+$, given an arbitrary $\varepsilon > 0$ we can find a δ satisfying $x > \delta > 0$ such that

$$f(u) = \sum_{k=0}^{m} \tfrac{f^{(k)}(x)}{k!}(u-x)^k + h_x(u)(u-x)^m, \ |u-x| \le \delta,$$

where $h_x(u)$ is a certain measurable function on $[x-\delta, x+\delta]$ satisfying the inequality $|h_x(u)| \le \varepsilon, |u-x| \le \delta$. Hence

$$(12) \quad T_\lambda^{(m)}(f;x) = \sum_{k=0}^{m} \tfrac{f^{(k)}(x)}{k!} \sum_{j=0}^{k} \tbinom{k}{j}(-1)^j x^j T_\lambda^{(m)}(u^{k-j};x)$$
$$+T_\lambda^{(m)}(h_x(u)(u-x)^{(m)}\chi_{\delta,x}(u);x) + T_\lambda^{(m)}(f\chi_{\delta,x}^c;x)$$
$$= \textstyle\sum_1 + \sum_2 + \sum_3, \text{ (say)}.$$

Using the fact that T_λ maps polynomials into polynomials, and the basic convergence theorem [3] we obtain

$$(13) \quad \textstyle\sum_1 = f^{(m)}(x) T_\lambda(u^m;1) \to f^{(m)}(x), \ \lambda \to \infty.$$

It follows from **Lemma1** that

$$T_\lambda^{(m)}(h_x(u)(u-x)^{(m)}\chi_{\delta,x}(u);x) = \tfrac{x^{\alpha-1}}{a(\lambda)} \sum_{k=0}^{m} \sum_{\upsilon=0}^{[\frac{m-k}{2}]} \lambda^{\upsilon+k} \int_{x-\delta}^{x+\delta} u^{-\alpha} h_x(u)(u-x)^m$$
$$G^{\lambda-m}(xu^{-1}) \left|G'(xu^{-1})\right|^k g_{\upsilon,k,m}(x,u)du.$$

The δ above can be chosen so small that

$$\left|G'(xu^{-1})\right| \le A\,|u-x|\,, \ |u-x| < \delta,$$

where A is some constant. Since the functions $g_\upsilon, k, m(x,u)$ are bounded on $[x-\delta, x+\delta]$, it is clear that there exists a constant M_1 independent of λ, ε and δ such that for all λ sufficiently large,

$$\left|T_\lambda^{(m)}(h_x(u)(u-x)^{(m)}\chi_{\delta,x}(u);x)\right| \le \varepsilon M_1 \sum_{k=0}^{m} \sum_{\upsilon=0}^{[\frac{m-k}{2}]} \lambda^{\upsilon+k} T_\lambda(|u-x|^{m+k};x)$$
$$\le \varepsilon M_2 \sum_{k=0}^{m} \sum_{\upsilon=0}^{[\frac{m-k}{2}]} \lambda^{\upsilon+k-\frac{m+k}{2}},$$

by [3] where M_2 is another constant not depending on λ, ε and δ. Since $\upsilon \leq [\frac{m-k}{2}], \upsilon + k - \frac{m+k}{2} - [\frac{m-k}{2}] - \frac{m-k}{2} \leq 0$, there exists a constant M independent of λ, ε and δ such that

(14) $|\sum_2| \leq M$, for all λ sufficiently large.

To estimate $\sum_3$, first of all we notice that there exists a positive integer p and a constant P such that

$$\left|\{G'(xu^{-1})\}^k g_{\upsilon,k,m}(x,u)\right| \leq P(1+u^{-p}), u \in IR^+$$

and $0 \leq k \leq m, 0 \leq \upsilon \leq [\frac{m-k}{2}]$. Hence by lemma 1, we have

$$|\sum_3| \leq P \sum_{k=0}^{m} \sum_{\upsilon=0}^{[\frac{m-k}{2}]} \lambda^{\upsilon+k} \frac{x^{\alpha-1}}{a(\lambda)} \int_0^\infty u^{-\alpha}(1+u^{-p}) G^{\lambda-m}(xu^{-1}) f(u) \chi^c_{\delta,x}(u) du$$

$$= P \sum_{k=0}^{m} \sum_{\upsilon=0}^{[\frac{m-k}{2}]} \lambda^{\upsilon+k} \frac{a(\lambda-m)}{a(\lambda)} T_{\lambda-m}(f\chi_{\delta,x}; x) + \frac{a^{**}(\lambda-m)}{x^p a(\lambda)} T^{**}_{\lambda-m}(f\chi^c_{\delta,x}; x),$$

where T^{**}_λ corresponds to the operator (1) with α replaced by $\alpha + p$ and $a^{**}(\lambda)$ refers to the $a(\lambda)$ for T^{**}_λ. We observe that

$$\lim_{\lambda\to\infty} \frac{a(\lambda-m)}{a(\lambda)} = |\{G(1)\}|^{-m}$$

$$= \lim_{\lambda\to\infty} \frac{a^{**}(\lambda-m)}{a^{**}(\lambda)}.$$

Also, by the definition of the operator, T_λ we have

$$\lim_{\lambda\to\infty} \lambda^{\upsilon+k} T_{\lambda-m}(f\chi^c_{\delta,x}; x) = \lim_{\lambda\to\infty} \lambda^{\upsilon+k} T^{**}_{\lambda-m}(f\chi^c_{\delta,x}; x) = 0.$$

It follows that $\sum_3 \to 0$ as $\lambda \to \infty$.In view of this fact and (12)- - (14), it follows that there exist a λ_0 such that

$$\left|T^{(m)}_\lambda(f;x) - f^{(m)}(x)\right| < (2+M)\varepsilon, \lambda > \lambda_0.$$

Since M does not depend on ε we have (11).

The uniformity part is easy to derive from the above proof by noting that, to begin with, δ can be chosen independent of $x \in [a,b]$ so that $|h_x(u)| \leq \varepsilon$ for $x \in [a,b]$ whenever $|u-x| \leq \delta$. Then, it is clear that the various constants occuring in the above proof can be chosen so as not to depend on $x \in [a,b]$.

This completes the proof of Theorem 5.

Finally, we show that the asymptotic formula of Theorem 2 remains valid in the pointwise simultaneous approximation as well. We observe that the difference between Theorem2 and the following one lies in the assumptions on f and G. We have

Theorem 7 *-: Let $m \in IN, G \in C^{(m)}_b(IR^+) \cap T(IR^+), G'''(1)$ exist and $G''(1)$ be non-zero. let Ω be any bounding function for G and $f \in D_\Omega$. Then*

(15) $T^{(m)}_\lambda(f;x) - f^{(m)}(x) = \frac{1}{2\lambda[G''(1)]^2}[f^{(m)}(x) m G(1)\{(2\alpha - m - 5)G''(1) - G'''(1)\}$

$$+ x f^{(m+1)}(x) G(1)\{2(\alpha - m - 3)G''(1) - G'''(1)\}$$

$$+x^2 f^{(m+2)}(x)G(1)G''(1)] + o(\tfrac{1}{\lambda}),\ (\lambda \to \infty).$$

Whenever $x \in IR^+$ is such that $f^{(m+2)}(x)$ exists. Also, if $f^{(m+2)}$ exists and is continuous on $< a, b >$, then (15) holds uniformly in

$x \in [a, b]$.

Proof. :If $f^{(m+2)}(x)$ exists, we have

$$f(u) = \sum_{k=0}^{m+2} \tfrac{f^{(k)}(x)}{k!}(u-x)^k + h(u,x),$$

where $h(u,x) \in D_\Omega$ and for any $\varepsilon > 0$, there exists a $\delta > 0$, such that $|h(u,x)| \le \varepsilon\, |u-x|^{m+2}$ for all sufficiently $|u-x| \le \delta$.

Thus

(16) $$T_\lambda^{(m)}(f;x) = T_\lambda^{(m)}(Q;x) + T_\lambda^{(m)}(h(u,x);x),$$

where $$Q = \sum_{k=0}^{m+2} \tfrac{f^{(k)}(x)}{k!}(u-x)^k \text{ is a polynomial in } u.$$

Clearly, $Q \in D_\Omega^{(m)}$ for $\Omega(u) = 1 + u^{m+2}$ which is bounding function for every $G \in T(IR^+)$. Also, $Q^{(k)}(x) = f^{(k)}(x)$, for $k = m, m+1, m+2$. Hence applying Theorem 2, we have

(17) $$T_\lambda^{(m)}(Q;x) = \tfrac{1}{2\lambda[G''(1)]^2}[f^{(m)}(x)mG(1)\{(2\alpha - m - 5)G''(1) - G'''(1)\}$$
$$+xf^{(m+1)}(x)G(1)\{2(\alpha - m - 3)G''(1) - G'''(1)\}$$
$$+x^2 f^{(m+2)}(x)G(1)G''(1)] + o(\tfrac{1}{\lambda}),\ (\lambda \to \infty)..$$

To establish (15), it remains to show that,

(18)$T_\lambda^{(m)}(h(u,x);x) = o(\frac{1}{\lambda}), (\lambda \to \infty)$.

For this we have by Lemma 1

$$\left|T_\lambda^{(m)}(h(u,x);x)\right| \le \tfrac{x^{\alpha-1}}{a(\lambda)} \sum_{k=0}^{m} \sum_{\upsilon=0}^{[\frac{m-k}{2}]} \lambda^{\upsilon+k} \int_0^\infty u^{-\alpha} G^\lambda(xu^{-1}) \left|G'(xu^{-1})\right|$$
$$g_{\upsilon,k,m}(x,u)\{h(u,x)\chi_{\delta,x}^c(u) + \varepsilon\, |u-x|^{m+2}\}du.$$

Proceeding as in the proof of Theorem 5, we find that the term corresponding to ε in the above is bounded by $\frac{\varepsilon M}{\lambda}$ for some M independent of ε or λ, and the $\chi_{\delta,x}^c$-term contributes only a $o(\frac{1}{\lambda})$ quantity (in fact $o(\frac{1}{\lambda^p})$ for an arbitrary $p > 0$).Then in view of arbitraryness of $\varepsilon > 0$, (18) follows.

Then uniformity part follows a remark similar to that made for the proof of the uniformity part of Theorem 5. This completes the proof of the theorem.

In the rest of the paper , we study the second problem.

4.Some Direct Theorems for Linear Combinations

In this section we give some direct theorems for the linear combinations of the operators T_λ. First, we give some definitions. The k-th moment $\mu_{\lambda,k}(x), k \in IN^0$(set of non-negative integers)of the operator T_λ is defined by

(19) $\mu_{\lambda,k}(x) = T_\lambda((u-x)^k; x) = x^k \tau_{\lambda,k}$ (say).

Clearly $\tau_{\lambda,k}$ does not depend on x.

Now, we first prove a lemma on the moments $\mu_{\lambda,k}$.

Lemma 8 *2- Let $G \in T_\infty(IR^+)$ and $k \in IN^0$. Then there exist constants $\gamma_{k,\upsilon}, \upsilon \geq [\frac{k+1}{2}]$ such that the following asymptotic expansion is valid:*

(20) $\tau_{\upsilon,k} = \sum_{\upsilon=[\frac{k+1}{2}]}^{\infty} \gamma_{k,\frac{\upsilon}{\lambda}} \upsilon,\ \lambda \to \infty.$

Proof. By the definition we have

$$\tau_{\lambda,k} = \frac{1}{a(\lambda)} \int_0^\infty s^{\alpha-k-2}(1-s)^k G^\lambda(s) ds.$$

Let $\frac{1}{3} < \gamma < \frac{1}{2}$. Then

$$\int_{1-\lambda^{-\gamma}}^{1+\lambda^{-\gamma}} s^{\alpha-k-2}(1-s)^k G^\lambda(s) ds$$

$$= \int_{1-\lambda^{-\gamma}}^{1+\lambda^{-\gamma}} s^{\alpha-k-2}(1-s)^k \exp\{\lambda \log[G(1) + \tfrac{(s-1)^2}{2!} G''(1)$$

$$+ ... + \tfrac{(s-1)^{2m}}{2m!} G^{(2m)}(1) + o((s-1)^{2m})]\} ds, (m \geq 2)$$

$$= G^\lambda(1) \int_{1-\lambda^{-\gamma}}^{1+\lambda^{-\gamma}} s^{\alpha-k-2}(1-s)^k \exp\{\lambda[(\tfrac{(s-1)^2 G''(1)}{2!G(1)} + ... + \tfrac{(s-1)^{2m}}{(2m)!}\tfrac{G^{(2m)}(1)}{G(1)}$$

$$+ o((s-1)^{2m})) - \tfrac{1}{2}(\tfrac{(s-1)^2}{2!}\tfrac{G''(1)}{G(1)} + ... + \tfrac{(s-1)^{2m}}{(2m)!}\tfrac{G^{(2m)}(1)}{G(1)}$$

$$+ o((s-1)^{2m}))^2 + ...]\} ds$$

$$= G^\lambda(1) \int_{1-\lambda^{-\gamma}}^{1+\lambda^{-\gamma}} s^{\alpha-k-2}(1-s)^k \exp(\lambda \tfrac{(s-1)^2 G''(1)}{2G(1)}) \exp\{[C_3(s-1)^3 + C_4(s-1)^4$$

$$+ ... + C_{2m}(s-1)^{2m} + o((s-1)^{2m})]\} ds,$$

($C_i's$ being constants depending on $G(1), G''(1), ..., G^{(2m)}(1)$)

$$= G^\lambda(1) \int_{1-\lambda^{-\gamma}}^{1+\lambda^{-\gamma}} s^{\alpha-k-2}(1-s)^k \exp(\lambda \tfrac{(s-1)^2 G''(1)}{2G(1)})(1$$

$$+\sum_{3\le 3i\le j\le[2m+\frac{i-1}{\gamma}]} b_{ij}\lambda^i(s-1)^j+o(\lambda^{1-2m\gamma}))ds,$$

($b'_{ij}s$ depending on C'_is)

$$=G^\lambda(1)\int\limits_{1-\lambda^{-\gamma}}^{1+\lambda^{-\gamma}}[\{\sum_{l=0}^{[2m-\frac{1}{\gamma}]}a_l(s-1)^{k+l}\}$$

$$\{1+\sum_{3\le 3i\le j\le[2m+\frac{i-1}{\gamma}]} b_{ij}\lambda^i(s-1)^j\}+o(\lambda^{1-(2m+k)\gamma})]\exp\{\lambda\frac{(s-1)^2G''(1)}{2G(1)}\}ds$$

$$=G^\lambda(1)\int\limits_{1-\lambda^{-\gamma}}^{1+\lambda^{-\gamma}}[\sum_{\substack{3\le 3i\le j\le[2m+\frac{i-1}{\gamma}]\\ 0\le l\le[2m-\frac{1}{\gamma}]}} d_{ijl}\lambda^i(s-1)^{j+k+l}+o(\lambda^{1-(2m+k)\gamma})]\exp\{\lambda\frac{(s-1)^2G''(1)}{2G(1)}\}ds$$

(where $d'_{ijl}s$ are certain constants depending on a'_ls and $b'_{ij}s$ and vanish if j+k+l is odd).

$$=2G^\lambda(1)\int\limits_{1-\lambda^{-\gamma}}^{1+\lambda^{-\gamma}}[\sum_{\substack{3\le 3i\le j\le[2m+\frac{i-1}{\gamma}]\\ 0\le l\le[2m-\frac{1}{\gamma}]}} d_{ijl}\lambda^i(s-1)^{j+k+l}+o(\lambda^{1-(2m+k)\gamma})]\exp\{\lambda\frac{(s-1)^2G''(1)}{2G(1)}\}ds.$$

Putting

$$\lambda\frac{(s-1)^2G''(1)}{2G(1)}=-t,\ s=1+\{-\frac{2tG(1)}{\lambda G''(1)}\}^{\frac{1}{2}}\quad\text{and } ds=\{-\frac{G(1)}{2\lambda G''(1)t}\}^{\frac{1}{2}}dt,$$

and the last expression simplifies to

$$2G^\lambda(1)\int\limits_0^{-\lambda}1-2\gamma\frac{G''(1)}{2G(1)}[\sum_{\substack{3\le 3i\le j\le[2m+\frac{i-1}{\gamma}]\\ 0\le l\le[2m-\frac{1}{\gamma}]}} d_{ijl}\lambda^i\{-\frac{2tG(1)}{\lambda G''(1)}\}^{[\frac{j+k+l+1}{2}]}+o(\lambda^{1-(2m+k)\gamma})]\cdot$$

$$\cdot\, e^{-t}\{-\frac{G(1)}{2\lambda G''(1)t}\}^{\frac{1}{2}}dt$$

(since d_{ijl} vanish when $j+k+l$ is odd)

$$=\frac{2^{\frac{1}{2}}G^{\lambda+\frac{1}{2}}(1)}{\{-\lambda G''(1)\}^{\frac{1}{2}}}[\int\limits_0^{-\lambda}1-2\gamma\frac{G''(1)}{2G(1)}\sum_{0\le 3i\le j\le[2m+\frac{i-1}{\gamma}]} d^*_{ijl}\lambda^{i-[\frac{j+k+l-1}{2}]}t^{[\frac{j+k+l+1}{2}]-\frac{1}{2}}e^{-t}dt$$

$$+o(\lambda^{1-(2m+k)\gamma+1-2\gamma})],$$

(where $d^*_{ijl}=d_{ijl}\{-\frac{2G(1)}{G''(1)}\}^{[\frac{j+k+l+1}{2}]}$)

$$=\frac{2^{\frac{1}{2}}G^{\lambda+\frac{1}{2}}(1)}{\{-\lambda G''(1)\}^{\frac{1}{2}}}[\sum_{\substack{0\le 3i\le j\le[2m+\frac{i-1}{\gamma}]\\ 0\le l\le[2m-\frac{1}{\gamma}]}} d^{**}_{ijl}\lambda^{i-[\frac{j+k+l-1}{2}]}+o(\lambda^{2-(2m+2+k)\gamma})],$$

where $d^{**}_{ijl}=d^{*}_{ijl}\Gamma(([\frac{j+k+l-1}{2}])^{\gamma}+\frac{1}{2})$ and we have made use of the fact that by enlarging the integral in above from 0 to ∞, we are only adding the terms in λ which decay exponentially and therefore can be absorbed in the o-term.

Next we analyse the expression

$$\int\limits_{(0,\infty)-(1-\lambda^{-\gamma},1+\lambda^{-\gamma})} s^{\alpha-k-2}(1-s)^k G^{\lambda}(s)ds = E(\lambda), (say).$$

we have for any positive integer p,

$$|E(\lambda)| \le \lambda^{\gamma p}\int\limits_0^{\infty} s^{\alpha-k-2}\left|1-s\right|^{k+p} G^{\lambda}(s)ds$$
$$= \lambda^{\gamma p} a^{**}(\lambda) T^{**}_{\lambda}(|u-1|^{k+p};1),$$

where T^{**}_{λ} and $a^{**}(\lambda)$ are the same as considered in the proof of Theorem 5. By making use of an estimate for the operators T^{**}_{λ} [3] we have

$$|E(\lambda)| \le A\lambda^{\gamma p-\frac{k+p}{2}} a^{**}(\lambda),$$

where A is certain constant not depending on λ.Again making use of the same estimate as above, for $a^{**}(\lambda)$, we have

$$G^{-\lambda}(1)\left|E(\lambda)\right| = o(\lambda^{\gamma p-\frac{k+p+1}{2}}).$$

Thus, choosing p such that

$$p > \frac{2(2m+2+k)}{1-2\gamma},$$

we have

$$\int\limits_0^{\infty} s^{\alpha-k-2}(1-s)^k G^{\lambda}(s)ds$$
$$= \frac{2^{\frac{1}{2}}G^{\lambda+\frac{1}{2}}(1)}{\{-\lambda G''(1)\}^{\frac{1}{2}}}[\sum_{\substack{0\le 3i\le j\le[2m+\frac{i-1}{\gamma}]\\ 0\le l\le[2m-\frac{1}{\gamma}]}} d^{**}_{ijl}\lambda^{i-[\frac{j+k+l-1}{2}]}+o(\lambda^{2-(2m+2+k)\gamma})].$$

Now, for all indices under consideration we have

$$[\frac{j+k+l+1}{2}]-i=[\frac{j-2i+k+l+1}{2}]\ge[\frac{k+1}{2}],$$

and since m could be chosen arbitrarily large, there exist constants $C_{k,\upsilon}, \upsilon\ge[\frac{k+1}{2}]$ such that we have the following asymptotic expansion

$$\int\limits_0^{\infty} s^{\alpha-k-2}(1-s)^k G^{\lambda}(s)ds = \{-\frac{2G(1)}{\lambda G''(1)}\}^{\frac{1}{2}} G^{\lambda}(1)\sum_{\upsilon=[\frac{k+1}{2}]}^{\infty}\frac{C_{k,\upsilon}}{\lambda^{\upsilon}}.$$

Noting that $C_{0,0}=1$, it follows that there exist constants $\gamma_{k,\upsilon}, \upsilon\ge[\frac{k+1}{2}]$ such that (20) holds. This completes the proof of Lemma 2.

For a $G\in T_{\infty}(IR^{+})$ and any fixed set of positive constants $\alpha_i, i=0,1,2,...,k$, following Rathore [8] the linear combination $T_{\lambda,k}$ of the operators $T_{\alpha_i\lambda}, i=0,1,2,...,k$ is defined by

(21) $$T_{\lambda,k}(f;x) = \frac{1}{\Delta}\begin{vmatrix} T_{\alpha_0\lambda}(f;x) & \alpha_0^{-1} & \alpha_0^{-2} & .... & \alpha_0^{-k} \\ T_{\alpha_1\lambda}(f;x) & \alpha_1^{-1} & \alpha_1^{-2} & .... & \alpha_1^{-k} \\ ..... & .... & .... & .... & .... \\ ..... & .... & .... & .... & .... \\ T_{\alpha_k\lambda}(f;x) & \alpha_k^{-1} & \alpha_k^{-2} & .... & \alpha_k^{-k} \end{vmatrix},$$

where Δ is the determinant obtained by replacing the operator column by the entries '1'. Clearly

(22) $$T_{\lambda,k} = \sum_{j=0}^{k} C(j,k) T_{\alpha_j\lambda},$$

for constants $C(j,k), j = 0,1,2,...,k$, which satisfy $\sum_{j=1}^{k} C(j,k) = 1$.

$T_{\lambda,k}$ is called a linear combination of order k. For k=0, $T_{\lambda,0}$ denotes the operator T_λ itself. We remark here that the above definition of linear combination $T_{\lambda,k}$ is dependent on the assumption that $G \in T_\infty(IR^+)$.That is to say, our results on the linear combinations $T_{\lambda,k}$ are not neccessarily valid if these conditions are violated.

Theorem 9 *7- Let $G \in T_\infty(IR^+), \Omega$ be a bounding function for G and $f \in D_\Omega$. If at a point $x \in IR^+, f^{(2k+2)}$ exists, then*

(23) $|T_{\lambda,k}(f;x) - f(x)| = O(\lambda^{-(k+1)})$,

(24) $|T_{\lambda,k+1}(f;x) - f(x)| = o(\lambda^{-(k+1)})$,

where $k = 0,1,2,.....$ Also, if $f^{(2k+2)}$ exists and is continuous on $< a,b >\subseteq IR^+$, (23)——(24) hold uniformly on $[a,b]$.

Proof.-First we show that if it is only assumed that $G \in T(IR^+)$ and $G''(1)$ exists and is non-zero, then

(25) $$T_{\lambda,k}(f;x) - f(x) = \sum_{j=1}^{2k+2} \frac{x^j f^{(j)}(x)}{j!}\tau_{\lambda,j} + o(\lambda^{-(k+1)}),$$

if $x \in IR^+$ is such that $f^{(2k+2)}(x)$ exists and $f \in D_\Omega$ for a certain bounding function Ω for G.

To prove (25) with the assumption on f, we have

$$f(u) - f(x) = \sum_{j=1}^{2k+2} \frac{f^{(j)}(x)}{j!}(u-x)^j + R_x(u), u \to x,$$

where $R_x(u) = o((u-x)^{(2k+2)}), u \to x$. It is clear from the definition of $\tau_{\lambda,j}$ that we only have to show that

(26) $T_\lambda(R_x(u);x) = o(\lambda^{-(k+1)})$.

Obviously, $R_x(u) \in D_\Omega$. Now, given an arbitrary $\varepsilon > 0$ we can choose a $\delta > 0$ such that

$|R_x(u)| \leq \varepsilon(u-x)^{2k+2}, |u-x| \leq \delta.$

Hence by using the basic properties of the operators T_λ, we note that the result follows.

In this case the uniformity part is obvious.

Now, if in addition it is assumed that $G \in T_\infty(IR^+)$, Lemma 2 and (25) imply that

$$(27)\quad T_\lambda(f;x) - f(x) = \sum_{j=1}^{2k+2} \frac{x^j f^{(j)}(x)}{j!} \sum_{\upsilon=[\frac{j+1}{2}]}^{k+1} \frac{\gamma_{j,\upsilon}}{\lambda^\upsilon} + o(\lambda^{-(k+1)}),$$

which, in the uniformity case holds uniformly in $x \in [a,b]$.

Since the coefficients $C(j,k)$ in (22) obviously satisfy the relation

$$(28)\quad \sum_{j=0}^{k} C(j,k)\alpha_j^{-p} = 0, p = 1,2,3,...,k,$$

in view of (27), (23)—-(24) are immediate and so is the uniformity part.

This completes the proof of Theorem 7.

In the same spirit we have,

Theorem 10 *8- Let $G \in T_\infty(IR^+), \Omega$ be a bounding function for G and $f \in D_\Omega$. If $0 \leq p \leq 2k+2$ and $f^{(p)}$ exists and is continuous on $< a,b > \subseteq IR^+$, for each $x \in [a,b]$ and all λ sufficiently large, then*

$$(29)\quad |T_{\lambda,k}(f;x) - f(x)| \leq \max[\frac{C}{\lambda^{\frac{p}{2}}}\omega(f^{(p)};\lambda^{-\frac{1}{2}}), \frac{C'}{\lambda^{k+1}}],$$

where $C = C(k)$ and $C\prime = C\prime(k,f)$ are constants and $\omega(f^{(p)};\delta)$ denotes the local modulus of continuity of $f^{(p)}$ on $< a,b >$.

Proof.-There exists a $\delta > 0$ such that $[a-\delta, b+\delta] \subset < a,b >$. It is clear that if $u \in < a,b >$, there exists an η lying between $x \in [a,b]$ and u such that

$$(30)\quad \left| f(u) - f(x) - \sum_{j=1}^{p} \frac{f^{(j)}(x)}{j!}(u-x)^j \right| \leq \frac{|u-x|^p}{p!}\left|f^{(p)}(\eta) - f^{(p)}(x)\right| \leq \frac{|u-x|^p}{p!}(1 + \frac{|u-x|}{\lambda^{-\frac{1}{2}}})\omega(f^{(p)};\lambda^{-\frac{1}{2}}),$$

using a well known reesult on modulus of continuity [11]. If the expression occuring within the modulus sign on the left hand side of the above inequality is denoted by $F_x(u)$, by a well known property of T_λ, it follows that

$$T_{\alpha_j\lambda}(F_x(u)\chi^c_{\delta,x}(u);x) = o(\lambda^{-(k+1)}),$$

uniformly in $x \in [a,b]$. By (30), we have

$$\left|T_{\alpha_j\lambda}(F_x(u)\chi^c_{\delta,x}(u);x)\right| \leq \frac{b^p}{p!}(A_p + A_{p-1})(\alpha_j\lambda)^{-\frac{p}{2}}\omega(f^{(p)};\lambda^{-\frac{1}{2}})$$

for all λ sufficiently large and $x \in [a,b]$. Here A_p, A_{p-1} are constants depending on p. Hence, for a constant C_p independent of f such that for all $x \in [a,b]$,

(32) $\left|T_{\lambda,k}(F_x(u)\chi^c_{\delta,x}(u);x)\right| \leq C_p\lambda^{-\frac{p}{2}}\omega(f^{(p)};\lambda^{-\frac{1}{2}})$.

Applying the result (23) for the functions $1, u, u^2, ..., u^p$, we find that there exists a constant C'' depending on

$\max\{|f'(x)|,...,\left|f^{(p)}(x)\right|; x \in [a,b]\}$ and p such that for all $x \in [a,b]$,

(33) $\left|T_{\lambda,k}(\sum_{j=1}^{p}\frac{f^{(j)}(x)}{j!}(u-x)^j;x)\right| \leq C''\lambda^{-(k+1)}$

Now, (29) is clear from (31)————(33). This completes the proof of Theorem 8.

Finally, we prove a result concerned with the degree of simultaneous approximation by the linear combinations $T_{\lambda,k}$.

Theorem 11 *9- Let $G \in C_b^{(m)}(IR^+) \cap T_\infty(IR^+), \Omega$ a bounding function for G and $f \in D_\Omega$. If at a point $x \in IR^+, f^{(2k+2+m)}$ exists, then*

(34) $\left|T^{(m)}_{\lambda,k}(f;x) - f^{(m)}(x)\right| = O(\lambda^{-(k+1)})$,

and

(35) $\left|T^{(m)}_{\lambda,k+1}(f;x) - f^{(m)}(x)\right| = o(\lambda^{-(k+1)})$,

where $k = 0, 1, 2, ...,$. also if $f^{(2k+2+m)}$ exists and is continuous on $< a, b > \subseteq IR^+$, (34)——-(35) hold uniformly in $x \in [a,b]$.

Proof.- If $f^{(2k+2+m)}(x)$ exists, we can find a neighbourhood $(a\prime, b\prime)$ of x such that $f^{(m)}$ exist and is continuous on $(a\prime, b\prime)$. Let $g(u)$ be an infinitely differentiable function with *supp* $g \subseteq (a\prime, b\prime)$ such that $g(u) = 1$, for $u \in [x-\delta, x+\delta]$ for some $\delta > 0$. then , an application of Lemma 1 shows that

(36) $T^{(m)}_{\lambda,k}(f(u) - f(u)g(u);x) = o(\lambda^{-(k+1)})$.

In the uniformity case, we consider a g with *supp* $g \subseteq < a, b >$ with $g(u) = 1$ for $u \in [a-\delta, b+\delta] \subseteq < a, b >$ and then (35) holds uniformly in $x \in [a,b]$.

Since $f(u)g(u) \in C_b^{(m)}(IR^+)$ we have

(37) $T^{(m)}_{\lambda}(fg;x) = x^{-m}T_\lambda(u^m\{f(u)g(u)\}^{(m)};x)$.

Now, since $u^m\{f(u)g(u)\}^{(m)}$ is $(2k+2)$-times differentiable at x(and continuously on $(a-\delta, b+\delta)$ in the uniformity case), applying Theorem 7, we have

(38) $\left|T^{(m)}_{\lambda,k}(fg;x) - f^{(m)}(x)\right| = O(\lambda^{-(k+1)})$,

and

(39) $\left|T^{(m)}_{\lambda,k+1}(fg;x) - f^{(m)}(x)\right| = o(\lambda^{-(k+1)})$,

where in the uniformity case these holds in $x \in [a,b]$. Thus, combining (36)—(39), we get (34)——(35). This completes the proof of the Theorem 9.

Now, we obtain a result which is the analogue of the Theorem 8, in the case of simultaneous approximation.

Theorem 12 *10- Let* $m \in IN, G \in C_b^{(m)}(IR^+) \cap T_\infty(IR^+), \Omega$ *a bounding function for* G *and* $f \in D_\Omega$. *If* $0 \leq p \leq 2k+2$ *and* $f^{(m+p)}$ *exists and is continuous on* $< a, b > \subseteq IR^+$ *for each* $x \in [a,b]$, *then, for all sufficiently large* λ,

(40) $$\left|T_{\lambda,k}^{(m)}(f;x) - f^{(m)}(x)\right| \leq \max\{\frac{C_m}{\lambda^{\frac{k}{2}}}\omega(f^{(p+m)};\lambda^{-\frac{1}{2}}), \frac{C'_m}{\lambda^{k+1}}\},$$

where $C_m = C_m(k), C'_m = C'_m(k,f)$ *are constants and* $\omega(f^{(p+m)};\delta)$ *denotes the local modulus of continuity of* $f^{(p+m)}$ *on* $,< a, b >$.

Proof.- The proof of this theorem follows from Lemma 1 and Theorems 5—–9.

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References

References

[1] R. A. DEVORE AND G. G. LORENTZ, Constructive Approximation. Springer-Verlag Berlin Heidelberg, New York, (1993).

[2] P.P. KOROVKIN, Linear Operators and Approximation Theory (1960), Delhi. (Translated from Russian edition of 1959).

[3] B. Kunwar, Approximation of analytic functions by a class of linear positive operators, J. Approx. Theory 44. 173-182 (1985).

[4] G. G. LORENTZ, Bernstein Polynomials, Toronto (1953).

[5] C. P. MAY, Saturation and inverse theorems for combination of a class of exponential type operators. Canad. J. Math. 28,(1976), 1224-50.

[6] M. W.MULLER, Die Folge der Gamma operatoren.Thesis,Technische Hoschule,Stuttgort.(1967).

[7] R. K. S. RATHORE, Linear combinations of linear positive operators and generating relations in special functions, Dissertation, IIT Delhi (India) (1973).

[8] R. K. S. RATHORE, Approximation of unbounded functions with Linear Positive Operators, Doctoral Dissertation, Technische Hogeschool Delft (1974).

[9] P. C. SIKKEMA AND R. K. S. RATHORE,Convolution with powers of bell shaped functions. Report, Dept. of Math. Technische Hogeschool (1976).

[10] P. C. SIKKEMA, Approximation formulae of Voronovskaya- type for certain convolution operators, J. Approx. Theory 26(1979), 26-45.

[11] A. F. TIMAN, Theory of Approximation of Functions of a Real variable, Peargamon Press (1963).

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