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$\varphi\psi$-CONTINUITY BETWEEN TWO POINTWISE QUASI-UNIFORMITY SPACES

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Abstract. In this paper, by means of operations (is called here φ, ψ) we shall define $\varphi\psi$-continuity between two pointwise quasi-uniform spaces and prove that the category of pointwise quasi-uniform spaces and pointwise quasi-uniformly continuous morphisms is topological.

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1. Introduction

There have been all kinds of studies about the theory of quasi-proximity and quasi-uniformity in fuzzy set theory (see [1-3, 5, 6,8-12, etc.]). Moreover the concept of S-quasi-uniformity was also generalized to [0, 1]-fuzzy set theory by Ghanim et al. [6].

In this paper, by means of operations φ, ψ we shall define $\varphi\psi$-continuity between pointwise quasi-uniform spaces and prove that the category **LOCDL-PQUnif** of pointwise quasi-uniform spaces and pointwise quasi-uniformly continuous morphisms is topological over **SET** $\times$ **LOCDL** with respect to the forgetful functor V : **LOCDL** $-$ **PQUnif** $\rightarrow$ **SET** $\times$ **LOCDL**, where the category **CDL** is the category having objects of completely distributive lattice and morphisms of complete lattice homomorphisms. **LOCDL** is the dual **CDL**^{op} of **CDL**.

2. Preliminaries

For a completely distributive lattice K [14, 15] and a nonempty set X, K^X denotes the set of all K-fuzzy sets on X. K^X is also a completely distributive lattice when it inherits the structure of lattice K in a natural way, by defining $\vee, \wedge, \leq$ pointwisely. The set of nonunit prime elements [7] in K^X is denoted by $P(K^X)$. The set of nonzero co-prime elements [7] in K^X is denoted by $M(K^X)$. A nonzero co-prime element in K^X is also called a molecule [16].

Definition 2.1. *(The categories CDL and LOCDL). The category* **CDL** *is the category having objects of completely distributive lattices and morphisms of complete lattice homomorphisms, i.e., those mappings which preserve arbitrary joins and arbitrary meets.* **LOCDL** *is the dual* $\mathbf{CDL}^{op}$ *of* **CDL**.

Definition 2.2. *(Rodabaugh [14]). Let* **C** *be a subcategory of* **LOQML**. *An object in the category* $\mathbf{SET} \times \mathbf{C}$ *is an ordered pair* (X, K) *, where* $X \in |\mathbf{SET}|, K \in |\mathbf{C}|$, *a morphism is an ordered pair* $(f, \phi) : (X, K) \to (Y, L)$, *where* $f : X \to Y$ *in* **SET**, $\phi : K \to L$ *in* **C**. *Composition and identity morphisms are defined component-wise in the respective categories.*

When $\mathbf{C} = \mathbf{LOCDL}$, we obtain the category $\mathbf{SET} \times \mathbf{LOCDL}$.

Definition 2.3. *(Shi [15]). Let* $(X, K) \in |\mathbf{SET} \times \mathbf{LOCDL}|$. *A pointwise quasi-uniformity on* K^X *(or on (X, K)) is a nonempty subset* $\mathcal{U}$ *of* $\mathcal{R}(K^X)$ *satisfying*
(U1) $u \in \mathcal{R}(K^X), v \in \mathcal{U}, u \leq v \Rightarrow u \in \mathcal{U}$.
(U2) $u, v \in \mathcal{U} \Rightarrow u \vee v \in \mathcal{U}$.
(U3) $u \in \mathcal{U} \Rightarrow \exists v \in \mathcal{U}$ such that $v \odot v \geq u$.

Let $\mathcal{U}$ be a pointwise quasi-uniformity on (X, K). Then $\mathcal{A} \subset \mathcal{U}$ is called a basis of $\mathcal{U}$ if for each $u \in \mathcal{U}$, there exists $v \in \mathcal{A}$ such that $u \leq v$. $\mathcal{B} \subset \mathcal{U}$ is called a sub-basis of $\mathcal{U}$ if the set of all finite suprema of elements of $\mathcal{B}$ is a basis of $\mathcal{U}$. When $\mathcal{U}$ is a poinwise quasi-uniformity on $(X, K), (X, K, \mathcal{U})$ or $(K^X, \mathcal{U})$ is called a pointwise quasi-uniform space.

Lemma 2.4. *Let* $\mathcal{R}(K^X)$ *denote the set of all R-mappings from* $M(K^X)$ *to* K^X, $R(L^Y)$ *denote the set of all R-mappings from* $M(L^Y)$ *to* L^Y *and* $(f, \phi) : (X, K) \to (Y, L)$ *be in* $\mathbf{SET} \times \mathbf{LOCDL}$. *Then for any* $v \in \mathcal{R}(L^Y), (f, \phi)^{\leftarrow} \circ v \circ (f, \phi)^{\to} \in \mathcal{R}(K^X)$. *In [15], the mapping* $(f, \phi)^{\leftarrow} \circ v \circ (f, \phi)^{\to}$ *from* $\mathcal{R}(L^Y)$ *to* $\mathcal{R}(K^X)$ *is denoted by* $(f, \phi)^* : \mathcal{R}(L^Y) \to \mathcal{R}(K^X)$.

As was proved in [15], we can obtain the following two theorems.

Theorem 2.5. *Let* $(f, \phi) : (X, K) \to (Y, L)$ *be in* $\mathbf{SET} \times \mathbf{LOCDL}$ *and* $\mathcal{V}$ *be a pointwise quasi-uniformity on* (Y, L). *Then* $(f, \phi)^*(\mathcal{V})$ *is a basis for a pointwise quasi-uniformity* $\mathcal{U}$ *on (X, K),* $\mathcal{U}$ *is written as* $(f, \phi)^{\leftarrow}(\mathcal{V})$ *[15].*

Theorem 2.6. *Let $(f,\phi) : (X,K) \to (Y,L)$ be in* **SET** $\times$ **LOCDL** *and $\mathcal{V}$ be a pointwise quasi-uniformity on (Y,L) Then $\eta((f,\phi)^{\leftarrow}(\mathcal{V})) = (f,\phi)^{\leftarrow}(\eta(\mathcal{V}))$ [15].*

Definition 2.7. *Let $(X,K,\mathcal{U})$ be a pointwise quasi-uniform space. A mapping $\varphi : K^X \to K^X$ is called an operation on K^X if for each $A \in L^X \setminus \emptyset, int(A) \leq A^{\varphi}$ and $\emptyset^{\varphi} = \emptyset$ where A^{φ} denotes the value of φ in A [4].*

3.Main Results

Definition 3.1. *Let $(X,K,\mathcal{U})$ and $(Y,L,\mathcal{V})$ be two pointwise quasi-uniform spaces. Let φ,ψ be two operations on K^X, L^Y respectively $(f,\phi) : (X,K) \to (Y,L)$ in* **SET**$\times$ **LOCDL** *is said to be pointwise quasi-uniformly $\varphi\psi$-continuous with respect to $\mathcal{U}$ and $\mathcal{V}$ if for each $v \in \mathcal{V}$, there exists $u \in \mathcal{U}$ such that for any $a,b \in M(K^X)$,*

$$b \text{ not} \leq u^{\varphi}(a) \Rightarrow (f,\phi)^{\rightarrow}(b) \text{ not} \leq v^{\psi}((f,\phi)^{\rightarrow}(a)).$$

Theorem 3.2. *Let $(X,K,\mathcal{U})$ and $(Y,L,\mathcal{V})$ are two pointwise quasi-uniform spaces. Let φ,ψ be two operations on K^X, L^Y respectively. Then for $(f,\phi) : (X,K) \to (Y,L)$ in* **SET** $\times$ **LOCDL***, the following statements are equivalent :*
(1) (f,ϕ) is pointwise quasi-uniformly $\varphi\psi$-continuous.
(2) For each $v \in \mathcal{V}$, there exists $u \in \mathcal{U}$ such that $(f,\phi)^{\leftarrow} \circ v^{\psi} \circ (f,\phi)^{\rightarrow} \leq u^{\psi}$.
(3) $(f,\phi)^{*}(\mathcal{V})^{\psi} \subset \mathcal{U}^{\varphi}$.

Proof. The proof is straightforward from Definition 3.5. and Lemma 2.4. □

Corollary 3.3. *Let φ be operation on K^X, and ψ be operation on L^Y and H^Z. If $(f,\phi) : (X,K,\mathcal{U}) \to (Y,L,\mathcal{V})$ is pointwise quasi-uniformly $\varphi\psi$-continuous and $(g,\psi) : (Y,L,\mathcal{V}) \to (Z,H,\mathcal{W})$ is pointwise quasi-uniformly $\varphi\psi$-continuous, then $(g,\psi) \circ (f,\phi) : (X,K,\mathcal{P}) \to (Z,H,\mathcal{R})$ is pointwise quasi-uniformly $\varphi\psi$-continuous.*

In Theorem 3.2, if we suppose that $\varphi = \psi = id$ then we shall have the following results:

(I) Let $(X,K,\mathcal{U})$ and $(Y,L,\mathcal{V})$ are two pointwise quasi-uniform spaces. Then for $(f,\phi) : (X,K) \to (Y,L)$ in **SET**$\times$**LOCDL**, the following statements are equivalent:
(1) (f,ϕ) is pointwise quasi-uniformly continuous.
(2) For each $v \in \mathcal{V}$, there exists $u \in \mathcal{U}$ such that $(f,\phi)^{\leftarrow} \circ v \circ (f,\phi)^{\rightarrow} \leq u$.
(3) (f, $\phi)^{*}(\mathcal{V}) \subset \mathcal{U}$.

(II) If both $(f,\phi) : (X,K,\mathcal{U}) \to (Y,L,\mathcal{V})$ and $(g,\rho) : (Y,L,\mathcal{V}) \to (Z,H,\mathcal{W})$ are pointwise quasi-uniformly continuous, then so is $(g,\rho) \circ (f,\phi) : (X,K,\mathcal{P}) \to (Z,H,\mathcal{R})$.

It is clear that the identical mapping $id_{(X,K)} : (X,K) \to (X,K)$ is pointwise quasi-uniformly continuous. Therefore from results (I) and (II) we get the following:

Theorem 3.4. *Pointwise quasi-uniform spaces and pointwise quasi-uniformly continuous morphisms form a category, called the category of pointwise quasi-uniform spaces, denoted by* **LOCDL** $\times$ **PQUnif**.

We shall prove that the category **LOCDL-PQUnif** is topological over **SET** $\times$ **LOCDL** with respect to the forgetful functor V : **LOCDL** – **PQUnif** $\to$ **SET** $\times$ **LOCDL**, where the forgetful functor V : **LOCDL** – **PQUnif** $\to$ **SET** $\times$ **LOCDL** has the following shape:

$$V(X,K,\mathcal{U}) = (\mathrm{X, K}), \quad V(f,\phi) = (f,\phi).$$

A V-structured source from pointwise quasi-uniform space looks like
$((X,K),(f_i,\phi_i) : (X,K) \to V(X_i,K_i,\mathcal{U}_i))_\Omega$.

Lemma 3.5. *Let* $((X,K),(f_i,\phi_i) : (X,K) \to V(X_i,K_i,\mathcal{U}_i))_\Omega$ *be a V-structured source in* **SET** $\times$ **LOCDL** *from* **LOCDL-PQUnif**. *then there exists a pointwise quasi-uniformity* $\mathcal{U}$ *on (X, K) such that each* $(f_i,\phi_i) : (X,K,\mathcal{U}) \to (\mathrm{X}_i,K_i,\mathcal{U}_i)$ *is pointwise quasi-uniformly* $\varphi\psi$*-continuous.*

Proof. Put

$$\mathcal{A} = \{(f_i,\phi_i)^*(\mathcal{U}_i^\psi)|i \in \Omega\} = \{(f_i,\phi_i)^{\leftarrow} \circ u_i^\psi \circ (f_i,\phi_i)^{\rightarrow}|u_i \in \mathcal{U}_i, i \in \Omega\}.$$

Then it is easy to show that $\mathcal{A}$ is a subbases for a pointwise quasi-uniformity $\mathcal{U}^\varphi$ on (X,K) and each (f_i,ϕ_i) is pointwise quasi-uniformly $\varphi\psi$-continuous. □

Lemma 3.6. $((X,K,\mathcal{U}),\ (f_i,\phi_i) : (X,K) \to V(X_i,K_i,\mathcal{U}_i))_\Omega$ *is an initial V-lift of the V-structured source of Lemma 3.5 where* $\mathcal{U}$ *is given in the proof of Lemma 3.5.*

Proof. Let $((X,K,\mathcal{V})),\ (g_i,\mu_i) : (X,K) \to V(X_i,K_i,\mathcal{U}_i))_\Omega$ be another V-lift of the V-structured source of Lemma 3.5, and let $(h,\rho) : (X,K) \to (X,K)$ be a ground morphism such that

$$\forall i \in \Omega,\ (g_i,\mu_i) = (f_i,\phi_i) \circ (h,\rho).$$

Then $(g_i,\mu_i)^{\leftarrow} = (h,\rho)^{\leftarrow} \circ (f_i,\phi_i)^{\leftarrow}$. To prove that $(h,\rho) : (X,K,\mathcal{V}) \to (X,K,\mathcal{U})$ is pointwise quasi-uniformly continuous, we shall prove that $(h,\rho) : (X,K,\mathcal{V}) \to (X,K,\mathcal{U})$ is pointwise quasi-uniformly $\varphi\psi$-continuous, take $u \in \mathcal{U}$. Then by the proof of Lemma 3.9 we know that there exists a finite family $\{\mathrm{i}_1, i_2, \cdots, i_n\} \subset \Omega$ and $\{u_{i_1} \in \mathcal{U}_{i_1}, u_{i_2} \in \mathcal{U}_{i_2}, \cdots, u_{i_n} \in \mathcal{U}_{i_n}\}$ such that

$$u^\psi \leq \bigvee_{i=1}^n (f_i,\phi_i)^{\leftarrow} \circ u_i^\psi \circ (f_i,\phi_i)^{\rightarrow}.$$

Hence

$(h,\rho)^{\leftarrow} \circ u^{\psi} \circ (h,\rho)^{\rightarrow} \leq (h,\rho)^{\leftarrow} \circ \left(\bigvee_{i=1}^{n} (f_i,\phi_i)^{\leftarrow} \circ u_i^{\psi} \circ (f_i,\phi_i)^{\rightarrow}\right) \circ (h,\rho)^{\rightarrow}$

$= \bigvee_{i=1}^{n}((h,\rho)^{\leftarrow} \circ (f_i,\phi_i)^{\leftarrow} \circ u_i^{\psi} \circ (f_i,\phi_i)^{\rightarrow} \circ (h,\rho)^{\rightarrow})$

$= \bigvee_{i=1}^{n}((g_i,\mu_i)^{\leftarrow} \circ u_i^{\psi} \circ (g_i,\mu_i)^{\rightarrow}) \leq v^{\varphi} \in \mathcal{V}^{\varphi}$. (For $v \in \mathcal{V}$)

This shows that $(h,\rho) : (X,K,\mathcal{V}) \to (X,K,\mathcal{U})$ is pointwise quasi-uniformity $\varphi\psi$-continuous. Now we are getting $\varphi = \psi = id$, so $(h,\rho) : (X,K,\mathcal{V}) \to (X,K,\mathcal{U})$ is pointwise quasi-uniformly continuous. □

Lemma 3.7. $((X,K,\mathcal{U}), (f_i,\phi_i) : (X,K) \to V(X_i,K_i,\mathcal{U}_i))_{\Omega}$ *is the unique V-initial lift of the V-structured source of Lemma 3.5, where* $\mathcal{U}$ *is given in the proof of Lemma 3.5.*

Proof. *Let* $((X,K,\mathcal{V}), (g_i,\mu_i) : (X,K) \to V(X_i,K_i,\mathcal{U}_i))_{\Omega}$ be another V-initial lift. Now we prove that $\mathcal{U} = \mathcal{V}$. Since both constructs are lifts, it follows immediately that

$(f_i,\phi_i) = V(f_i,\phi_i) = V(g_i,\mu_i) = (g_i,\mu_i)$.

Let $(h,\rho) : (X,K) \to (X,K)$ by $h = id_X, \quad \rho = id_K$. Then using the initial properties of each lift, we have (h,ρ) must be pointwise quasi-uniformly $\varphi\psi$-continuous, so with getting $\varphi = \psi = id$ we have (h,ρ) is pointwise quasi-uniformly continuous, both from $(X,K,\mathcal{V})$ to $X,K,\mathcal{U})$, and from $(X,K,\mathcal{U})$ to $(X,K,\mathcal{V})$; the former implies $\mathcal{U} \subset \mathcal{V}$, and the latter implies $\mathcal{V} \subset \mathcal{U}$. So $\mathcal{U} = \mathcal{V}$. □

Theorem 3.8 *The category* **LOCDL-PQUnif** *is topological over* **SET** $\times$ **LOCDL** *with respect to the forgetful functor* V : **LOCDL − PQUnif** $\to$ **SET** $\times$ **LOCDL**.

Proof. With conjoining Lemma 3.5, 3.6, 3.7 and interpretation of Definition 1.3.1 in [14], we can obtain the proof. □

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