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ON CERTAIN CLASSES OF ANALYTIC FUNCTIONS DEFINED BY MEANS OF A LINEAR OPERATOR

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Abstract. In this paper, we introduce certain classes of analytic functions in the unit disk. The object of the present paper is to derive some interesting properties of functions belonging to these classes.

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1.*INTRODUTION*

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n \ z^n, \tag{1.1}$$

which are analytic in the unit disc $E = \{z : z \in \mathbb{C}, |z| < 1\}$.Let the functions f_i bqe defined for $i = 1, 2$,by

$$f_i(z) = z + \sum_{n=2}^{\infty} a_{n,i} \ z^n, \tag{1.2}$$

The modified Hadamard product(convolution) of f_1 and f_2 is defined here by

$$(f_1 * f_2)(z) = z + \sum_{n=2}^{\infty} a_{n,1} a_{n,2} \ z^n.$$

Let $P_k(\rho)$ be the class of functions $h(z)$ analytic in E satisfying the properties $h(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{\mathrm{Re} h(z) - \rho}{1 - \rho} \right| d\theta \leq k\pi, \tag{1.3}$$

where $z = re^{i\theta}, k \geq 2$ and $0 \leq \rho < 1$. This class has been introduced in [10]. We note, for $\rho = 0$, we obtain the class P_k defined and studied in [11], and for $\rho = 0, k = 2$, we have the well-known class P of functions with positive real part. The case $k = 2$ gives the class $P(\rho)$ of functions with positive real part greater than ρ. From (1.3) we can easily deduce that $h \in P_k(\rho)$ if and only if, there exists $h_1, h_2 \in P(\rho)$ such that for $z \in E$,

$$h(z) = \left(\frac{k}{4} + \frac{1}{2}\right) h_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) h_2(z). \tag{1.4}$$

where $h_i(z) \in P(\rho)$, $i = 1, 2$ and $z \in E$.

We have the following classes

$$R_k(\alpha) = \left\{ f : f \in A \text{ and } \frac{zf'(z)}{f(z)} \in P_k(\alpha), z \in E, 0 \leq \alpha < 1 \right\},$$

we note that $R_2(\alpha) = S^*(\alpha)$ is the class of starlike functions of order α.

$$V_k(\alpha) = \left\{ f : f \in A \text{ and } \frac{(zf'(z))'}{f'(z)} \in P_k(\alpha), z \in E, 0 \leq \alpha < 1 \right\}.$$

Note that $V_2(\alpha) = C(\alpha)$ is the class of convex functions of order α.

$$\begin{aligned} T_k(\rho, \alpha) &= \left\{ f : f \in A, g \in R_2(\alpha) \text{ and } \frac{zf'(z)}{f(z)} \in P_k(\rho), z \in E, 0 \leq \alpha, \rho < 1 \right\}. \\ T_k^*(\rho, \alpha) &= \left\{ f : f \in A, g \in V_2(\alpha) \text{ and } \frac{(zf'(z))'}{f(z)} \in P_k(\rho), z \in E, 0 \leq \alpha, \rho < 1 \right\}. \end{aligned}$$

In particular, the class $T_2^*(\rho, \alpha) = C(\rho, \alpha)$ was introduced by Noor [8] and for $T_2^*(0, 0) = C^*$ is the class of quasi-convex univalent functions which was first introduced and studied in [7].

It is obvious from the above definition that

$$f(z) \in V_k(\alpha) \iff zf'(z) \in R_k(\alpha), \tag{1.5}$$

$$f(z) \in T_k^*(\rho, \alpha) \iff zf'(z) \in T_k(\rho, \alpha). \tag{1.6}$$

Let $f \in A$.Denote $D^\lambda : A \longrightarrow A$ the operator defined by

$$D^\lambda f(z) = \frac{z}{(1 - z)^{\lambda+1}} * f(z), \qquad (\lambda > -1).$$

It is obvious that $D^0 f(z) = f(z)$, $D^1 f(z) = zf'(z)$ and

$$D^k f(z) = \frac{z(z^{k-1} f(z))^{(k)}}{k!}, \quad k \in \mathbb{N}_0 = \{0, 1, 2, ...\}.$$

The operator $D^k f(z)$ is called the kth order Ruscheweyh derivative of f.Recently Noor [6] and Noor [9] defined and studied an integral operator $I_n : A \longrightarrow A$ analogous to $D^k f$ as follows.

Let $f_n(z) = \frac{z}{(1-z)^{n+1}}$, $n \in \mathbb{N}_0$ and let $f_n^{(-1)}$ be defined such that

$$f_n(z) * f_n^{(-1)}(z) = \frac{z}{(1-z)^2}.$$

Then

$$I_n = f_n^{(-1)}(z) * f = \left[\frac{z}{(1-z)^{n+1}}\right]^{(-1)} * f.$$

We note that $I_0 f(z) = zf'(z)$ and $I_1 f(z) = f(z)$.The operator I_n is called the Noor integral operator of nth order, see [2, 5].

For any complex numbers a, b, c other than $0, -1, -2...$ the hypergeometric series is defined by

$${}_2F_1(a, b; c; z) = 1 + \frac{ab}{c1!} z + \frac{a(a+1)b(b+1)}{c(c+1)2!} z^2 + ... \tag{1.7}$$

We note that the series (1.7) converges absolutely for all z so that it represents an analytic function in E.Also an incomplete beta function $\phi(a, c, z)$ is related to the Gauss hypergeometric function ${}_2F_1(a, b; c; z)$ as

$${}_2F_1(1, b; c; z) = \phi(a, c, z),$$

and we note that $\phi(2, 1, z) = \frac{z}{(1-z)^a}$, where $\phi(2, 1, z)$ is the Koebe function. Using $\phi(a, c, z)$ a convolution operator, see [1] was defined by Carlson and Shaffer.We introduce a function $(z\ {}_2F_1(a, b; c; z))^{(-1)}$ given by

$$(z\ {}_2F_1(a, b; c; z)) * (z {}_2F_1(a, b; c; z))^{(-1)} = \frac{z}{(1-z)^{\lambda+1}}, \quad (\lambda > -1),$$

and obtain the following linear operator:

$$I_\lambda(a, b, c) f(z) = (z\ {}_2F_1(a, b; c; z))^{(-1)} * f(z), \tag{1.8}$$

where a, b, c are real numbers other $0, -1, -2, -3, ...$, $\lambda > -1$, $z \in E$ and $f \in A$. The operator I_λ is known as the generalized Noor integral operator. In particular, with

$b = 1$, the operator was studied in [3] for p-valent functions.By some computation we note that

$$I_\lambda(a,b,c)f(z) = z + \sum_{k=1}^{\infty} \frac{(c)_k(\lambda+1)_k}{(a)_k(b)_k(1)_k} a_{k+1} z^{k=1}, \tag{1.9}$$

where $(x)_k$ is the Pochhammer symbol defined by $(x)_k = x(x+1)...(x+k-1)$, $k = 1, 2, ...$ and $(x)_k = 1$, $k = 0$.

From (1.8), we note that

$$I_\lambda(a, \lambda+1, a)f(z) = f(z), \quad I_\lambda(a, 1, a)f(z) = zf'(z).$$

Also it can easily be verified that

$$z(I_\lambda(a,b,c)f(z))' = (\lambda+1)I_{\lambda+1}(a,b,c)f(z) - \lambda I_\lambda(a,b,c)f(z). \tag{1.10}$$

$$z(I_\lambda(a+1,b,c)f(z))' = aI_\lambda(a,b,c)f(z) - (a-1)I_\lambda(a+1,b,c)f(z). \tag{1.11}$$

We define the following subclasses.

Definition 1.1. *Let $f \in A$.Then $f \in R_k(a,b,c,\lambda,\alpha)$ if and only if $I_\lambda(a,b,c)f(z) \in R_k(\alpha)$, for $z \in E$.*

Definition 1.2. *Let $f \in A$.Then $f \in V_k(a,b,c,\lambda,\alpha)$ if and only if $I_\lambda(a,b,c)f(z) \in V_k(\alpha)$, for $z \in E$.*

Definition 1.4. *Let $f \in A$.Then $f \in T_k(a,b,c,\lambda,\rho,\alpha)$ if and only if $I_\lambda(a,b,c)f(z) \in T_k(\rho,\alpha)$, for $z \in E$.*

Definition 1.5. *Let $f \in A$.Then $f \in T_k^*(a,b,c,\lambda,\rho,\alpha)$ if and only if $I_\lambda(a,b,c)f(z) \in T_k^*(\rho,\alpha)$, for $z \in E$.*

We shall .need the following result.

Lemma 1.1 [4].Let $u = u_1 + iu_2$, $v = v_1 + iv_2$ and $\Psi(u,v)$ be a complex valued function satisfying the conditions: (i). $\Psi(u,v)$ is continuous in a domain $D \subset \mathbb{C}^2$, (ii). $(1,0) \in D$ and $\operatorname{Re}\Psi(1,0) > 0$, (iii). $\operatorname{Re}\Psi(iu_2, v_1) \le 0$, whenever $(iu_2, v_1) \in D$ and $v_1 \le -\frac{1}{2}\left(1+u_2^2\right)$.
If $h(z) = 1 + c_1 z + \cdots$ is a function analytic in E such that $(h(z), zh'(z)) \in D$ and $\operatorname{Re}\{\Psi(h(z), zh'(z))\} > 0$ for $z \in E$, then $\operatorname{Re} h(z) > 0$ in E.

2.Main results

Theorem 2.1. *Let $f \in A$. Then*

$$R_k(a,b,c,\lambda+1,\alpha) \subset R_k(a,b,c,\lambda,\alpha),$$

where α is given by

$$\alpha = \frac{2}{(2\lambda+1) + \sqrt{(2\lambda+1)^2+8}}. \tag{2.1}$$

Proof. Let $f \in R_k(a,b,c,\lambda+1,\alpha)$ and let

$$\frac{z(I_\lambda(a,b,c)f(z))'}{I_\lambda(a,b,c)f(z)} = p(z) = (\frac{k}{4}+\frac{1}{2})p_1(z) - (\frac{k}{4}-\frac{1}{2})p_2(z). \tag{2.2}$$

Then $p(z)$ is analytic in E with $p(0)=1$. Some computation and use of (1.10) yields

$$\frac{z(I_{\lambda+1}(a,b,c)f(z))'}{I_{\lambda+1}(a,b,c)f(z)} = \left\{p(z) + \frac{zp'(z)}{p(z)+\lambda}\right\} \in P_k, \quad z \in E.$$

Let

$$\Phi_\lambda(z) = \sum_{j=1}^{\infty} \frac{\lambda+j}{\lambda+1} z^j = \frac{\lambda}{\lambda+1}\frac{z}{(1-z)} + \frac{1}{\lambda+1}\frac{z}{(1-z)^2}.$$

Then

$$\begin{aligned} p(z) * \Phi_\lambda(z) &= p(z) + \frac{zp'(z)}{p(z)+\lambda}. \\ &= (\frac{k}{4}+\frac{1}{2})\{p_1(z)*\Phi_\lambda(z)\} - (\frac{k}{4}-\frac{1}{2})\{p_2(z)*\Phi_\lambda(z)\} \\ &= (\frac{k}{4}+\frac{1}{2})\left[p_1(z) + \frac{zp_1'(z)}{p_1(z)+\lambda}\right] - (\frac{k}{4}-\frac{1}{2})\left[p_2(z) + \frac{zp_2'(z)}{p_2(z)+\lambda}\right], \end{aligned}$$

and implies that

$$\left[p_i(z) + \frac{zp_i'(z)}{p_i(z)+\lambda}\right] \in P, \quad i=1,2, \ z \in E.$$

We want to show that $p_i(z) \in P(\alpha)$, where α is given by (2.1) and this will show that $p \in P_k$ for $z \in E$. Let

$$p_i(z) = (1-\alpha)h_i(z) + \alpha, \quad i = 1,2.$$

Then

$$\left[(1-\alpha)h_i(z) + \alpha + \frac{(1-\alpha)zh_i'(z)}{h_i(z)+\alpha+\lambda}\right] \in P.$$

We form the functional $\Psi(u,v)$ by choosing $u = h_i(z), \ v = zh_i'(z)$.

$$\Psi(u,v) = \left\{(1-\alpha)u + \alpha + \frac{(1-\alpha)v}{(1-\alpha)u+\alpha+\lambda}\right\}.$$

The first two conditions of Lemma 1.1 are clearly satisfied.We verify the condition (iii) as follows.

$$\mathrm{Re}\{\Psi(iu_2, v_1)\} = \alpha + \left\{\frac{(1-\alpha)(\alpha+\lambda)v_1}{(\alpha+\lambda)^2 + (1-\alpha)^2u_2^2}\right\}.$$

By putting $v \le -\frac{(1+u_2^2)}{2}$, we obtain

$$\begin{aligned}\mathrm{Re}\{\Psi(iu_2^2, v_1)\} &\le \alpha - \frac{1}{2}\left\{\frac{(1-\alpha)(\alpha+\lambda)(1+u_2^2)}{(\alpha+\lambda)^2+(1-\alpha)^2u_2^2}\right\}. \\ &= \frac{2\alpha(\alpha+\lambda)^2+2\alpha(1-\alpha)^2u_2^2-(1-\alpha)(\alpha+\lambda)-(1-\alpha)(\alpha+\lambda)u_2^2}{2\{(\alpha+\lambda)^2+(1-\alpha)^2u_2^2\}} \\ &= \frac{A+Bu_2^2}{2C},\end{aligned}$$

where

$$\begin{aligned}A &= 2\alpha(\alpha+\lambda)^2-(1-\alpha)(\alpha+\lambda), \\ B &= 2\alpha(1-\alpha)^2-(1-\alpha)(\alpha+\lambda), \\ C &= (\alpha+\lambda)^2+(1-\alpha)^2u_2^2 > 0.\end{aligned}$$

We notic that $\mathrm{Re}\{\Psi(iu_2^2, v_1)\} \le 0$ if and only if $A \le 0$, $B \le 0$. From $A \le 0$, we obtain α as given by (2.1) and $B \le 0$ gives us $0 \le \rho < 1$. Therefore applying Lemma 1.1, $h_i \in P$, $i = 1, 2$ and consequently $h \in P_k(\rho)$ for $z \in E$.This completes the proof.

Theorem 2.2. *For* $\lambda > -1$,

$$V_k(a,b,c,\lambda+1,0) \subset V_k(a,b,c,\lambda,\alpha),$$

where α *is given by* (2.1).

Proof. Let $f \in V_k(a,b,c,\lambda+1,0)$.Then $I_{\lambda+1}(a,b,c)f(z) \in V_k(0) = V_k$ and by (1.5), $z(I_{\lambda+1}(a,b,c)f(z))' \in R_k(0) = R_k$.This implies that $I_{\lambda+1}(a,b,c)(zf'(z) \in R_k \Rightarrow zf'(z) \in R_k(a,b,c,\lambda+1,0) \subset R_k(a,b,c,\lambda,\alpha)$. Consequently $f \in V_k(a,b,c,\lambda,\alpha)$, where α is given by (2.1).

Theorem 2.3. *Let* $\lambda > -1$. *Then*

$$T_k(a,b,c,\lambda+1,\rho,0) \subset T_k(a,b,c,\lambda,\gamma,\alpha),$$

where α *is given by* (2.1) *and* $\gamma \le \rho$ *is defined in the proof.*

Proof. Let $f \in T_k(a,b,c,\lambda+1,\rho,0)$.Then there exist $g \in R_2(a,b,c,\lambda+1,\rho,0)$ such that

$$\frac{z(I_{\lambda+1}(a,b,c)f(z))'}{I_{\lambda+1}(a,b,c)g(z)} \in P_k(\rho), \text{ for } z \in E, \quad 0 \le \rho < 1.$$

Let

$$\begin{aligned}\frac{z(I_\lambda(a,b,c)f(z))'}{I_\lambda(a,b,c)g(z)} &= (1-\gamma)p(z)+\gamma \\ &= (\frac{k}{4}+\frac{1}{2})\{(1-\gamma)p_1(z)+\gamma\} - (\frac{k}{4}-\frac{1}{2})\{(1-\gamma)p_2(z)+\gamma\},\end{aligned}$$

where $p(0) = 1$, and $p(z)$ is analytic in E.Making use of (1.10) and Theorem 2.1 with $k = 2$, we have

$$\begin{aligned} & \left\{ \frac{z(I_{\lambda+1}(a,b,c)f(z))'}{I_{\lambda+1}(a,b,c)g(z)} - \rho \right\} \\ = & \left\{ (1-\gamma)p(z) + (\gamma - \rho) + \frac{(1-\gamma)zp'(z)}{(1-\alpha)q(z)+\alpha+\lambda} \right\} \in P_k, \end{aligned} \tag{2.3}$$

and $q \in P$, where

$$(1-\alpha)q(z) + \alpha = \frac{z(I_\lambda(a,b,c)g(z))'}{I_\lambda(a,b,c)g(z)}, \quad z \in E.$$

Using (1.4) we form the functional $\varphi(u,v)$ by taking $u = u_1 + iu_2 = p_i(z)$, $v = v_1 + iv_2 = zp_i'(z)$ in (2.3) as

$$\varphi(u,v) = (1-\gamma)u + (\gamma-\rho) + \frac{(1-\gamma)v}{(1-\alpha)q(z)+\alpha+\lambda}. \tag{2.4}$$

It can be easily seen that the function $\varphi(u,v)$ defined by (2.4) satisfies the conditions (i) and (ii) of Lemma 1.1.To verify the condition (iii), we proceed with $q(z) = q_1 + iq_2$, as follows;

$$\begin{aligned} \operatorname{Re}\{\varphi(iu_2, v_1)\} &= (\gamma-\rho) + \operatorname{Re}\left\{ \frac{(1-\gamma)v_1}{(1-\alpha)(q_1+iq_2)+\alpha+\lambda} \right\} \\ &= (\gamma-\rho) + \frac{(1-\gamma)(1-\alpha)v_1q_1 + (1-\gamma)(\alpha+\lambda)v_1}{[(1-\alpha)q_1+\alpha+\lambda]^2 + (1-\alpha)^2q_2^2} \\ &= (\gamma-\rho) - \frac{1}{2}\frac{(1-\gamma)(1-\alpha)(1+u_2^2)q_1 + (1-\gamma)(\alpha+\lambda)(1+u_2^2)}{[(1-\alpha)q_1+\alpha+\lambda]^2 + (1-\alpha)^2q_2^2} \leq 0, \ \gamma \leq \rho < 1. \end{aligned}$$

Therefore applying Lemma 1.1, $p_i \in P$, $i = 1, 2$ and consequently $p \in P_k$ and thus $f \in T_k(a,b,c,\lambda,\gamma,\alpha)$.

Using the same technique and relation (1.6) with Theorem 2.3, we have the following result.

Theorem 2.4. *For $\lambda > -1$,*

$$T_k^*(a,b,c,\lambda+1,\rho,\alpha) \subset T_k^*(a,b,c,\lambda,\rho,\alpha),$$

where γ and α are given in Theorem 2.3.

We note that for different choices of parameters a, b, c, k and λ we obtain several intersting special cases for the result proved in this paper.

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