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# ON CERTAIN CLASS OF MEROMORPHIC FUNCTIONS DEFINED BY MEANS OF A LINEAR OPERATOR

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Dedicated to my Supervisor Professor Dr. Khalida Inayat Noor on the occasion of getting award of Noor integral operator organized by COMSATS Institute of Information Technology, H-8/1 Islamabad, Pakistan.

Abstract. The purpose of the present paper is to introduce new class $MB(\alpha, \lambda, q, s, A, B)$ of meromorphic functions defined by using a meromorphic analogue of the Choi-Saigo-Srivastava operator for the generalized hypergeometric function and investigate a number of inclusion relationships and radius problem of this class.The subordination relations, distortion theorems, and inequality properties are discussed by applying differential subordination method.

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## 1. Intoduction

Let $M$ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k \ z^k, \tag{1.1}$$

which are analytic in the punctured unit disk

$$E^* = \{z \ : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = E \backslash \{0\}.$$

If $f$ and $g$ are analytic in $E = E \cup \{0\}$, we say that $f$ is subordinate to $g$, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function $w$ in $E$ such that $f(z) = g(w(z))$.

For a complex parameters $\alpha_1, ...\alpha_q$ and $\beta_1, ...\beta_s$ $\quad (\beta_j \in \mathbb{C} \backslash \mathbb{Z}_0^- = \{0, -1, -2, ...\}$; $j = 1, ...s)$, we now define the generalized hypergeometric function

$${}_qF_s(\alpha_1, ...\alpha_q; \beta_1, ...\beta_s) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k ...(\alpha_q)_k}{(\beta_1)_k ...(\beta_q)_k \quad k!} z^k, \tag{1.2}$$

$(q \leq s+1; s \in \mathbb{N} \cup \{0\}; \mathbb{N} = \{1, 2, ...\}; z \in E)$, where $(v)_k$ is the Pochhammer symbol(or shifted factorial) defined in (terms of the Gamma function) by

$$(v)_k = \frac{\Gamma(v+k)}{\Gamma(v)} = \begin{cases} 1 & \text{if } k = 0 \text{ and } v \in \mathbb{C}\backslash\{0\} \\ v(v+1)...(v+k-1) & \text{if } k \in \mathbb{N} \text{ and } v \in \mathbb{C}. \end{cases}$$

Corresponding to a function

$$(\alpha_1, ...\alpha_q; \beta_1, ...\beta_s; z) = z^{-1} \ {}_qF_s(\alpha_1, ...\alpha_q; \beta_1, ...\beta_s; z). \tag{1.3}$$

Liu and Srivastava [8] consider a linear operator

$H(\alpha_1, ...\alpha_q; \beta_1, ...\beta_s) : M \longrightarrow M$ defined by the following Hadamard product(or convolution):

$$H(\alpha_1, ..\alpha_q; \beta_1, ..\beta_s)f(z) = (\alpha_1, ...\alpha_q; \beta_1, ...\beta_s; z) * f(z). \tag{1.4}$$

We note that the linear operator $H(\alpha_1, ..\alpha_q; \beta_1, ..\beta_s)$ was motivated essentially by Dzoik and Srivastava [2]. Some interesting developments with the generalized hypergeometric function were considered recently by Dzoik and Srivastava [3, 4] and Liu and Srivastava [6, 7].

Corresponding to the function $(\alpha_1, ...\alpha_q; \beta_1, ...\beta_s; z)$ defined by (1.3), we introduce a function ${}_\lambda(\alpha_1, ...\alpha_q; \beta_1, ...\beta_s; z)$ given by

$$(\alpha_1, ...\alpha_q; \beta_1, ...\beta_s; z) * {}_\lambda(\alpha_1, ...\alpha_q; \beta_1, ...\beta_s; z) = \frac{1}{z(1-z)^\lambda} \quad (\lambda > 0). \tag{1.5}$$

Analogous to $H(\alpha_1, ..\alpha_q; \beta_1, ..\beta_s)$ defined by (1.4), we now define the linear operator $H_\lambda(\alpha_1, ..\alpha_q; \beta_1, ..\beta_s)$ on $M$ as follows:

$$H_\lambda(\alpha_1, ..\alpha_q; \beta_1, ..\beta_s)f(z) = {}_\lambda(\alpha_1, ...\alpha_q; \beta_1, ...\beta_s; z) * f(z) \tag{1.6}$$

$(\alpha_i, \beta_j \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ i = 1, ..q;\ j = 1, ..s;\ \lambda > 0; z \in E^*; f \in M)$.

For convenience, we write

$$H_{\lambda,q,s}(\alpha_1) = H_\lambda(\alpha_1, ..\alpha_q; \beta_1, ..\beta_s).$$

It is easily verified from the definition (1.5) and (1.6) that

$$z(H_{\lambda,q,s}(\alpha_1+1)f(z))' = \alpha_1 H_{\lambda,q,s}(\alpha_1)f(z) - (\alpha_1+1)H_{\lambda,q,s}(\alpha_1+1)f(z), \tag{1.7}$$

and

$$z(H_{\lambda,q,s}(\alpha_1)f(z))' = \lambda H_{\lambda+1,q,s}(\alpha_1)f(z) - (\lambda+1)H_{\lambda,q,s}(\alpha_1)f(z). \tag{1.8}$$

We note that the operator $(H_{\lambda,q,s}(\alpha_1)$ is closely related to the Choi-Saigo-Srivastava operator [1] for analytic functions, which includes the integral operator studied by Liu [5] and Noor et al [12, 13].

Now by using the operator $(H_{\lambda,q,s}(\alpha_1)$,we introduce some new class of meromorphic functions.

**Definition 1.1.** *Assume that $\mu > 0, \alpha \geq 0,\ -1 \leq B \leq 1,\ A \neq B,\ A \in \mathbb{R}$, we say that a function $f(z) \in M$ is in the class $MB(\alpha, \lambda, q, s, A, B)$ if it satisfies:*

$$(1-\alpha)(z(H_{\lambda,q,s}(\alpha_1)f(z))^{\mu}+\alpha z((H_{\lambda+1,q,s}(\alpha_1)f(z))(z(H_{\lambda,q,s}(\alpha_1)f(z))^{\mu-1} \prec \frac{1+Az}{1+Bz},$$

*$z \in E$. In particular, we let $MB(\alpha, \lambda, q, s, 1-2\rho, -1) \equiv MB(\alpha, \lambda, q, s, \rho)$ denote the subclass of $MB(\alpha, \lambda, q, s, A, B)$ for $A = 1-2\rho,\ B = -1$ and $0 \leq \rho < 1$. It is obvious that $f \in MB(\alpha, \lambda, q, s, \rho)$ if and only if $f \in M$ and satisfies*

$$(1-\alpha)(z(H_{\lambda,q,s}(\alpha_1)f(z))^{\mu}+\alpha z((H_{\lambda+1,q,s}(\alpha_1)f(z))(z(H_{\lambda,q,s}(\alpha_1)f(z))^{\mu-1} > \rho, \quad z \in E.$$

In this paper, we will discuss the subordination relations, inclusion relations, distortion theorems and inequalities properties of $MB(\alpha, \lambda, q, s, A, B)$.

## 2. Preliminary results

To establish our main results we need the following Lemma.

**Lemma 2.1** [10, 11]. *Let the function $h(z)$ be analytic and convex (univalent) in $E$ with $h(0) = 1$. Suppose also that the function $\Phi(z)$ given by*

$$\Phi(z) = 1 + c_1 z + c_2 z^2 + \ldots$$

*is analytic in $E$. If*

$$\Phi(z) + \frac{z\ \Phi'(z)}{\gamma} \prec h(z)\ \ (z \in E;\ \mathrm{Re}\gamma \geq 0;\ \gamma \neq 0)\,, \tag{2.1}$$

*then*

$$\Phi(z) \prec \Psi(z) = \frac{\gamma}{z^{\gamma}} \int_0^z t^{\gamma-1} h(t)dt \prec h(z)\ \ (z \in E)\,,$$

*and $\Psi(z)$ is the best dominant of* (2.1).

## 3.Main resuls

**Theorem 3.1.** *Let* $\mu > 0, \alpha \geq 0, -1 \leq B \leq 1, A \in \mathbb{R}, f(z) \in MB(\alpha, \lambda, q, s, A, B)$. *Then*

$$(zH_{\lambda,q,s}(\alpha_1)f(z))^{\mu} \prec \frac{\mu\lambda}{\alpha}\int_0^1 \frac{1+Azu}{1+Bzu}u^{\frac{\mu\lambda}{\alpha}-1}du \prec \frac{1+Az}{1+Bz}.$$

*Proof.* Consider the function $\phi(z)$ defined by

$$(zH_{\lambda,q,s}(\alpha_1)f(z))^{\mu} = \phi(z), \quad z \in E. \tag{3.1}$$

Then $\phi(z)$ is analytic in $E$ with $\phi(0) = 1$.Differentiating (3.1) with respect to $z$ and using the identity (1.8) in (3.1), we have

$$\left[\begin{array}{c} (1-\alpha)(z(H_{\lambda,q,s}(\alpha_1)f(z))^{\mu} + \alpha z((H_{\lambda+1,q,s}(\alpha_1)f(z))(z(H_{\lambda,q,s}(\alpha_1)f(z))^{\mu-1} \\ = \phi(z) + \frac{\alpha z\phi^{'}(z)}{\mu\lambda} \prec \frac{1+Az}{1+Bz},\ z \in E. \end{array}\right.$$

Now by using Lemma 2.1 for $\gamma = \frac{\mu\lambda}{\alpha}$, we deduce that

$$\begin{aligned}(zH_{\lambda,q,s}(\alpha_1)f(z))^{\mu} &= \phi(z) \prec \frac{\mu\lambda}{\alpha}z^{-\frac{\mu\lambda}{\alpha}}\int_0^z \frac{1+At}{1+Bt}t^{\frac{\mu\lambda}{\alpha}-1}dt \\ &= \frac{\mu\lambda}{\alpha}\int_0^1 \frac{1+Azu}{1+Bzu}u^{\frac{\mu\lambda}{\alpha}-1} \prec \frac{1+Az}{1+Bz}.\end{aligned}$$

**Corollary 3.2.***Let* $\mu > 0, \alpha \geq 0, \rho \neq 1$.*If*

$$(1-\alpha)(z(H_{\lambda,q,s}(\alpha_1)f(z))^{\mu}+\alpha z((H_{\lambda+1,q,s}(\alpha_1)f(z))(z(H_{\lambda,q,s}(\alpha_1)f(z))^{\mu-1} \prec \frac{1+(1-2\rho)z}{1-z},$$

$z \in E$, *then*

$$zH_{\lambda,q,s}(\alpha_1)f(z))^{\mu} \prec \rho + \frac{(1-\rho)\lambda\mu}{\alpha}\int_0^1 \frac{1+zu}{1-zu}u^{\frac{\mu\lambda}{\alpha}-1}du, \quad z \in E.$$

**Corollary 3.3.***Let* $\mu > 0, \alpha \geq 0$, *then*

$$MB(\alpha, \lambda, q, s, A, B) \subset MB(0, \lambda, q, s, A, B).$$

**Theorem 3.3.** *Let* $f \in MB(0, \lambda, q, s, \rho)$ *for* $z \in E$*.Then* $f \in MB(\alpha, \lambda, q, s, \rho)$ *for* $|z| < R(\alpha, \lambda, \mu)$*, where*

$$R(\alpha, \lambda, \mu) = \frac{\lambda\mu}{\alpha + \sqrt{\alpha^2 + \lambda^2\mu^2}}. \tag{3.2}$$

*Proof.* Set

$$zH_{\lambda,q,s}(\alpha_1)f(z))^{\mu} = (1-\rho)h(z) + \rho, \quad z \in E.$$

Now proceeding as Theorem 3.1, we have

$$(1-\alpha)(z(H_{\lambda,q,s}(\alpha_1)f(z))^{\mu} + \alpha z((H_{\lambda+1,q,s}(\alpha_1)f(z))(z(H_{\lambda,q,s}(\alpha_1)f(z))^{\mu-1}$$

$$= (1-\rho)\left\{h(z) + \frac{\alpha}{\lambda\mu} zh'(z)\right\}. \tag{3.3}$$

Using the following well known estimate [9]

$$\left|zh'(z)\right| \leq \frac{2r}{1-r^2} \operatorname{Re}\{h(z)\}, \quad (|z| = r < 1)$$

in (3.3), we get

$$\frac{\left\{(1-\alpha)(z(H_{\lambda,q,s}(\alpha_1)f(z))^{\mu} + \alpha z((H_{\lambda+1,q,s}(\alpha_1)f(z))(z(H_{\lambda,q,s}(\alpha_1)f(z))^{\mu-1} - \rho\right\}}{1-\rho}$$

$$= \operatorname{Re}\left\{h(z) + \frac{\alpha}{\lambda\mu} zh'(z)\right\} \geq \operatorname{Re}\left\{h(z) + \frac{\alpha}{\lambda\mu}\left|zh'(z)\right|\right\}$$

$$\geq \operatorname{Re} h(z)\left\{1 - \frac{2\alpha r}{\lambda\mu(1-r^2)}\right\}.$$

The right hand side of this inequality is positive if $r < R(\alpha, \lambda, \mu)$, where $R(\alpha, \lambda, \mu)$ is given by (3.2). Consequently it follows from (3.3) that $f \in MB(\alpha, \lambda, q, s, \rho)$ for $|z| < R(\alpha, \lambda, \mu)$.Sharpness of this result follows by taking $h_i(z) = \frac{1+z}{1-z}$, $i = 1, 2$ in (3.3).

**Theorem 3.3.** *Let* $0 \leq \alpha_2 \leq \alpha_{1.}$ *Then*

$$MB(\alpha_1, \lambda, q, s, A, B) \subset MB(\alpha_2, \lambda, q, s, A, B).$$

*Proof.* Let $f(z) \in MB(\alpha_1, \lambda, q, s, A, B)$.Then by Theorem 3.1 we have $f(z) \in MB(0, \lambda, q, s, A, B)$.

$$\left\{(1-\alpha_2)(z(H_{\lambda,q,s}(\alpha_1)f(z))^{\mu} + \alpha_2 z((H_{\lambda+1,q,s}(\alpha_1)f(z))(z(H_{\lambda,q,s}(\alpha_1)f(z))^{\mu-1}\right\}$$

$$= \frac{\alpha_2}{\alpha_1}\left\{(1-\alpha_1)(z(H_{\lambda,q,s}(\alpha_1)f(z))^{\mu} + \alpha_1 z((H_{\lambda+1,q,s}(\alpha_1)f(z))(z(H_{\lambda,q,s}(\alpha_1)f(z))^{\mu-1}\right\}$$
$$+(1-\frac{\alpha_2}{\alpha_1})(zH_{\lambda,q,s}(\alpha_1)f(z))^{\mu} \prec \frac{1+Az}{1+Bz}.$$

We see that$f(z) \in MB(\alpha_2, \lambda, q, s, A, B)$.

**Corollary 3.4.** *Let* $0 \leq \alpha_2 \leq \alpha_1$, $0 \leq \rho_1 \leq \rho_2$. *Then*

$$MB(\alpha_1, \lambda, q, s, \rho_2) \subset MB(\alpha_2, \lambda, q, s, \rho_1).$$

**Theorem 3.5.** *Let* $\mu > 0, \alpha \geq 0, -1 \leq B < A \leq 1,\ f(z) \in MB(\alpha, \lambda, q, s, A, B)$. *Then*

$$\frac{\lambda\mu}{\alpha}\int_0^1 \frac{1-Au}{1-Bu} u^{\frac{\lambda\mu}{\alpha}-1} du < \mathrm{Re}\,(zH_{\lambda,q,s}(\alpha_1)f(z))^{\mu}$$
$$< \frac{\lambda\mu}{\alpha}\int_0^1 \frac{1+Au}{1+Bu} u^{\frac{\lambda\mu}{\alpha}-1} du,\ z \in E, \tag{3.4}$$

*and the inequality* (3.4) *is sharp, with the extremal function defined by*

$$H_{\lambda,q,s}(\alpha_1)f(z) = z^{-1}\left\{\frac{\lambda\mu}{\alpha}\int_0^1 \frac{1+Azu}{1+Bzu} u^{\frac{\lambda\mu}{\alpha}-1} du\right\}^{\frac{1}{\mu}} \tag{3.5}$$

*Proof.* Since $f(z) \in MB(\alpha, \lambda, q, s, A, B)$, according to Theorem 3.1 we have

$$(zH_{\lambda,q,s}(\alpha_1)f(z))^{\mu} \prec \frac{\mu\lambda}{\alpha}\int_0^1 \frac{1+Azu}{1+Bzu} u^{\frac{\mu\lambda}{\alpha}-1} du \prec \frac{1+Az}{1+Bz}.$$

Therefore it follows from the definition of subordination and $A > B$ that

$$\begin{aligned}
\mathrm{Re}(zH_{\lambda,q,s}(\alpha_1)f(z))^{\mu} &< \sup_{z\in E}\mathrm{Re}\left\{\frac{\lambda\mu}{\alpha}\int_0^1 \frac{1+Azu}{1+Bzu} u^{\frac{\lambda\mu}{\alpha}-1} du\right\} \\
&\leq \frac{\lambda\mu}{\alpha}\int_0^1 \sup_{z\in E}\mathrm{Re}\left\{\frac{1+Azu}{1+Bzu}\right\} u^{\frac{\lambda\mu}{\alpha}-1} du \\
&< \frac{\lambda\mu}{\alpha}\int_0^1 \frac{1+Au}{1+Bu} u^{\frac{\lambda\mu}{\alpha}-1} du.
\end{aligned}$$

Also

$$\begin{aligned}\operatorname{Re}(zH_{\lambda,q,s}(\alpha_1)f(z))^{\mu} &> \inf_{z\in E}\operatorname{Re}\left\{\frac{\lambda\mu}{\alpha}\int_0^1\frac{1+Azu}{1+Bzu}u^{\frac{\lambda\mu}{\alpha}-1}du\right\}\\ &\geq \frac{\lambda\mu}{\alpha}\int_0^1\inf_{z\in E}\operatorname{Re}\left\{\frac{1+Azu}{1+Bzu}\right\}u^{\frac{\lambda\mu}{\alpha}-1}du\\ &> \frac{\lambda\mu}{\alpha}\int_0^1\frac{1-Au}{1-Bu}u^{\frac{\lambda\mu}{\alpha}-1}du.\end{aligned}$$

Note that the function $H_{\lambda,q,s}(\alpha_1)f(z)$ defined by (3.5) belongs to the class $MB(\alpha,\lambda,$ $q,s,A,B)$, we obtain the inequality (3.4) is sharp.Now by using the lines of proof of Theorem 3.5 we have the following results.

**Theorem 3.6.** *Let $\mu>0,\alpha\geq 0,-1\leq A<B\leq 1,$ $f(z)\in MB(\alpha,\lambda,q,s,A,B)$. Then*

$$\begin{aligned}\frac{\lambda\mu}{\alpha}\int_0^1\frac{1+Au}{1+Bu}u^{\frac{\lambda\mu}{\alpha}-1}du &< \operatorname{Re}\left(zH_{\lambda,q,s}(\alpha_1)f(z)\right)^{\mu}\\ &< \frac{\lambda\mu}{\alpha}\int_0^1\frac{1-Au}{1-Bu}u^{\frac{\lambda\mu}{\alpha}-1}du,\ z\in E,\end{aligned}\tag{3.6}$$

*and the inequality* (3.6) *is sharp, with the extremal function defined by* (3.5).

**Corollary 3.7.** *Let $\mu>0,\alpha\geq 0,0\leq\rho<1,$ $f(z)\in MB(\alpha,\lambda,q,s,A,B)$. Then*

$$\begin{aligned}\frac{\lambda\mu}{\alpha}\int_0^1\frac{1-(1-2\rho)u}{1+u}u^{\frac{\lambda\mu}{\alpha}-1}du &< \operatorname{Re}\left(zH_{\lambda,q,s}(\alpha_1)f(z)\right)^{\mu}\\ &< \frac{\lambda\mu}{\alpha}\int_0^1\frac{1+(1-2\rho)u}{1-u}u^{\frac{\lambda\mu}{\alpha}-1}du,\ z\in E,\end{aligned}\tag{3.7}$$

*and inequality* (3.7) *is equivalent to*

$$\begin{aligned}\rho+\frac{(1-\rho)\lambda\mu}{\alpha}\int_0^1\frac{1-u}{1+u}u^{\frac{\lambda\mu}{\alpha}-1}du &< \operatorname{Re}\left(zH_{\lambda,q,s}(\alpha_1)f(z)\right)^{\mu}\\ &< \rho+\frac{(1-\rho)\lambda\mu}{\alpha}\int_0^1\frac{1+u}{1-u}u^{\frac{\lambda\mu}{\alpha}-1}du,\ z\in E.\end{aligned}$$

**Theorem 3.8**.*Let* $\mu > 0, \alpha \geq 0, -1 \leq B < A \leq 1,\ f(z) \in MB(\alpha, \lambda, q, s, A, B)$. *Then*

$$\left\{ \frac{\lambda\mu}{\alpha} \int\limits_0^1 \frac{1-Au}{1-Bu} u^{\frac{\lambda\mu}{\alpha}-1} du \right\}^{\frac{1}{2}} < \mathrm{Re}\left(zH_{\lambda,q,s}(\alpha_1)f(z)\right)^{\frac{\mu}{2}}$$

$$< \left\{ \frac{\lambda\mu}{\alpha} \int\limits_0^1 \frac{1+Au}{1+Bu} u^{\frac{\lambda\mu}{\alpha}-1} du, \right\}^{\frac{1}{2}} \quad z \in E, \tag{3.8}$$

*and the inequality* (3.8) *is sharp with the extremal function defined by equation* (3.5).

*Proof.* By Theorem 3.1 we have

$$\{(zH_{\lambda,q,s}(\alpha_1)f(z))^{\mu}\} \prec \frac{1+Az}{1+Bz}.$$

Since $-1 \leq B < A \leq 1$, we have

$$0 \leq \frac{1-A}{1-B} < \{(zH_{\lambda,q,s}(\alpha_1)f(z))^{\mu}\} < \frac{1+A}{1+B}.$$

Hence the result follows by Theorem 3.5.

Note that the function $H_{\lambda,q,s}(\alpha_1)f(z)$ defined by (3.5) belongs to the class $MB(\alpha, \lambda, q, s, A, B)$, we obtain that the inequality (3.8) is sharp.Now by using the lines of proof of Theorem 3.8 we have the following result.

**Theorem 3.9.***Let* $\mu > 0, \alpha \geq 0, -1 \leq A < B \leq 1,\ f(z) \in MB(\alpha, \lambda, q, s, A, B)$. *Then*

$$\left\{ \frac{\lambda\mu}{\alpha} \int\limits_0^1 \frac{1+Au}{1+Bu} u^{\frac{\lambda\mu}{\alpha}-1} du \right\}^{\frac{1}{2}} < \mathrm{Re}\left(zH_{\lambda,q,s}(\alpha_1)f(z)\right)^{\frac{\mu}{2}}$$

$$< \left\{ \frac{\lambda\mu}{\alpha} \int\limits_0^1 \frac{1-Au}{1-Bu} u^{\frac{\lambda\mu}{\alpha}-1} du, \right\}^{\frac{1}{2}} \quad z \in E, \tag{3.9}$$

*and the inequality* (3.9) *is sharp, with the extremal function defined by* (3.5).

**Theorem 3.10.***Let* $\mu > 0, \alpha \geq 0, -1 \leq B < A \leq 1,\ f(z) \in MB(\alpha, \lambda, q, s, A, B)$.

**(i)** *If* $\alpha = 0$, *when* $|z| = r < 1$, *we have*

$$r^{-1}\left(\frac{1-Ar}{1-Br}\right)^{\frac{1}{\mu}} \leq |H_{\lambda,q,s}f(z)| \leq r^{-1}\left(\frac{1+Ar}{1+Br}\right)^{\frac{1}{\mu}} \tag{3.10}$$

*and inequality* (3.10) *is sharp , with the extremal function defined by*

$$H_{\lambda,q,s}(\alpha_1)f(z) = z^{-1}\left(\frac{1+Az}{1+Bz}\right)^{\frac{1}{\mu}}. \tag{3.11}$$

**(ii)** *If* $\alpha \neq 0$*, when* $|z| = r < 1$*, we havee*

$$r^{-1}\left(\frac{\lambda\mu}{\alpha}\int_0^1 \frac{1-Aru}{1-Bru}u^{\frac{\lambda\mu}{\alpha}-1}du\right)^{\frac{1}{\mu}} \leq |H_{\lambda,q,s}f(z)|$$

$$\leq r^{-1}\left(\frac{\lambda\mu}{\alpha}\int_0^1 \frac{1+Aru}{1+Bru}u^{\frac{\lambda\mu}{\alpha}-1}du\right)^{\frac{1}{\mu}}, \quad z \in E, \tag{3.12}$$

*and inequality* (3.12) *is sharp with the extremal function defined by* (3.5).

*Proof.* (i) If $\alpha = 0$. Since $f(z) \in MB(\alpha, \lambda, q, s, A, B)$, $-1 \leq B < A \leq 1$, we obtain from the definition of $MB(\alpha, \lambda, q, s, A, B)$ that

$$(zH_{\lambda,q,s}(\alpha_1)f(z))^{\mu} \prec \frac{1+Az}{1+Bz}.$$

Therefore it follows from the definition of the subordination that

$$(zH_{\lambda,q,s}(\alpha_1)f(z))^{\mu} = \frac{1+Aw(z)}{1+Bw(z)},$$

where $w(z) = c_1z + c_2z^2 + \ldots$ is analytic $E$ and $|w(z)| < |z|$, so when $|z| = r < 1$, we have

$$|(zH_{\lambda,q,s}(\alpha_1)f(z))|^{\mu} = \left|\frac{1+Aw(z)}{1+Bw(z)}\right| \leq \frac{1+A|w(z)|}{1+B|w(z)|} \leq \frac{1+Ar}{1+Br},$$

and

$$|(zH_{\lambda,q,s}(\alpha_1)f(z))|^{\mu} \geq \mathrm{Re}\,(zH_{\lambda,q,s}(\alpha_1)f(z))^{\mu} \geq \frac{1-Ar}{1-Br}.$$

It is obvious that (3.10) is sharp, with the extremal function defined by (3.11).

(ii) If $\alpha \neq 0$. according to Theorem 3.1 we have

$$(zH_{\lambda,q,s}(\alpha_1)f(z))^{\mu} \prec \frac{\lambda\mu}{\alpha}\int_0^1 \frac{1+Azu}{1+Bzu}u^{\frac{\lambda\mu}{\alpha}-1}du \prec \frac{1+Az}{1+Bz}.$$

Therefore it follows from the definition of the subordination

$$(zH_{\lambda,q,s}(\alpha_1)f(z))^{\mu} = \frac{\lambda\mu}{\alpha}\int_0^1 \frac{1+Aw(z)u}{1+Bw(z)u}u^{\frac{\lambda\mu}{\alpha}-1}du,$$

where $w(z) = c_1 z + c_2 z^2 + \ldots$ is analytic $E$ and $|w(z)| \le |z|$, so when $|z| = r < 1$, we have

$$\begin{aligned}
|(zH_{\lambda,q,s}(\alpha_1)f(z))|^{\mu} &\le \frac{\lambda\mu}{\alpha}\int_0^1 \left|\frac{1+Aw(z)u}{1+Bw(z)u}\right| u^{\frac{\lambda\mu}{\alpha}-1}du \\
&\le \frac{\lambda\mu}{\alpha}\int_0^1 \frac{1+Au\,|w(z)|}{1+Bu\,|w(z)|}u^{\frac{\lambda\mu}{\alpha}-1}du \\
&\le \frac{\lambda\mu}{\alpha}\int_0^1 \frac{1+Aur}{1+Bur}u^{\frac{\lambda\mu}{\alpha}-1}du,
\end{aligned}$$

and

$$|(zH_{\lambda,q,s}(\alpha_1)f(z))|^{\mu} \ge \mathrm{Re}\,(zH_{\lambda,q,s}(\alpha_1)f(z))^{\mu} \ge \frac{\lambda\mu}{\alpha}\int_0^1 \frac{1-Aur}{1-Bur}u^{\frac{\lambda\mu}{\alpha}-1}du.$$

Note that the function defined by (3.5) belongs to the class $MB(\alpha, \lambda, q, s, A, B)$, we obtain that the inequality (3.12) is sharp.By applying the techniques that we used in proving Theorem 3.10 we have the following theorem.

**Theorem 3.11.** *Let* $\mu > 0, \alpha \ge 0, -1 \le A < B \le 1$, $f(z) \in MB(\alpha, \lambda, q, s, A, B)$.

**(i)** *If* $\alpha = 0$, *when* $|z| = r < 1$, *we have*

$$r^{-1}\left(\frac{1+Ar}{1+Br}\right)^{\frac{1}{\mu}} \le |H_{\lambda,q,s}f(z)| \le r^{-1}\left(\frac{1-Ar}{1-Br}\right)^{\frac{1}{\mu}} \qquad (3.13)$$

*and inequality* (3.13) *is sharp , with the extremal function defined by* (3.11).

**(ii)** *If* $\alpha \ne 0$, *when* $|z| = r < 1$, *we have*

$$r^{-1}\left(\frac{\lambda\mu}{\alpha}\int_0^1 \frac{1+Aru}{1+Bru}u^{\frac{\lambda\mu}{\alpha}-1}du\right)^{\frac{1}{\mu}} \le |H_{\lambda,q,s}f(z)|$$

$$\leq r^{-1}\left(\frac{\lambda\mu}{\alpha}\int_0^1 \frac{1-Aru}{1-Bru}u^{\frac{\lambda\mu}{\alpha}-1}du\right)^{\frac{1}{\mu}}, \quad z \in E, \tag{3.14}$$

*and inequality* (3.14) *is sharp with the extremal function defined by* (3.5).

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