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EXISTENCE THEOREM FOR FREDHOLM TYPE INTEGRAL EQUATIONS WITH MODIFIED ARGUMENT

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Abstract In this paper some existence theorems for a Fredholm integral equation, are given by using nonexpansive mapping technique in $L^2(\Omega)$.

1. Introduction

In the paper [2], the author studies a Fredholm type integral equation of the form

$$y(x) = f(x) + \lambda \int_{\Omega} K(x, s, y(s))ds, \tag{1}$$

in $L^2(\Omega)$ using Contraction mapping principle. The purpose of this paper is to establish an existence theorems for a kind of Fredholm integral equation, where the mapping associated to equation is nonexpansive. Let $\Omega \subset \mathbb{R}^n$ be a measurable set. A function $f : \Omega \to \mathbb{R}$ Lebesgue measurable is called **square summabile**, if

$$\int_{\Omega} f^2(x)dx < \infty$$

Denote by $L^2(\Omega, \mathbb{R})$ the square summabile real functions set. It is well known that $L^2(\Omega, \mathbb{R})$ with norm $\|y\|_{L^2(\Omega)} = \left(\int_{\Omega} f^2(x)dx\right)^{1/2}$
is uniformly convex Banach space.

Theorem 1. (Browder-Ghode-Kirk) *Let X be a uniformly convex Banach space and $Y \subset X$, Y nonempty, bounded, closed and convex subset. If $f : Y \to Y$ be an nonexpansive map, then f has a fixed point.*

In paper [2], the author studies the solvability of equation (1.1) in $L^2(\Omega)$ by using the contraction mapping principle. Suppose that
$K : \overline{\Omega} \times \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ satisfies Caratheodory condition, that means:

(a) $f(\bullet,\bullet,u)$ is measurable in $\Omega\times\Omega$, for any $u\in\mathbb{R}$

(b) $f(t,s,\bullet)$ is continuous in $\mathbb{R}$, for almost all $t\in\Omega$ and $s\in\Omega$.

Theorem 2. (I.A.Rus,[2]) *Suppose that*

(i) $f\in L^2(\Omega)$ *and there exists* $L\in L^2(\Omega\times\Omega)$ *such that* $|K(x,s,u)-K(x,s,v)|\leq L(x,s)\cdot|u-v|$ *for any* $x,s\in\Omega$ *and* $u,v\in\mathbb{R}$;

(ii) $\int\limits_{\Omega}K(\bullet,s,0)\,ds\in L^2(\Omega)$;

(iii) $|\lambda|<\frac{1}{||L||_{L^2(\Omega\times\Omega)}}$;

In this conditions, the equation (1.1) has a solution in $L^2(\Omega)$ *which is obtained by succesive approximation method started from any element on* $L^2(\Omega)$.

2. Main results

We extend the Rus theorem, using the technique of nonexpansive mappings instead of the technique of Picard operators. We study the solvability of Fredholm integral equation with modified argument in $L^2(\Omega)$:

$$x(t)=f(t)+\int\limits_{\Omega}K(t,s,x(s),x(g(s)))ds, \tag{2}$$

where
$K:\overline{\Omega}\times\overline{\Omega}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$;
$f:\overline{\Omega}\to\mathbb{R}$;
$g:\overline{\Omega}\to\overline{\Omega}$.

Denote $\overline{B}(f,r)=\{x\in L^2(\Omega)/\,||x-f||_{L^2(\Omega)}\leq r, r>0\}\subset L^2(\Omega)$

Theorem 3. *Assume that the following conditions are satisfied:*

(i) $K(t,s,x,y)$ *satisfied the Caratheodory condition, i.e*

(a) $K(t,s,\bullet,\bullet)$ *is continuous almost everywhere in* $\mathbb{R}^2$ *for almost all* $t\in\Omega$ *and* $s\in\Omega$;

(b) $K(\bullet,\bullet,x,y)$ *is measurable in* $\overline{\Omega}\times\overline{\Omega}$, *for any* $(x,y)\in\mathbb{R}^2$ *fixed.*

(ii) $f\in L^2(\Omega)$ *and there exists* $L\in L^2(\Omega\times\Omega)$ *such that*

$$|K(t,s,u_1,u_2)-K(t,s,v_1,v_2)|\leq L(t,s)\cdot(|u_1-v_1|+|u_2-v_2|),$$

for any $t,s\in\Omega$ *and* $u_i,v_i\in\mathbb{R}, i=1,2$.

(iii) $||L||_{L^2(\Omega\times\Omega)}\cdot mes(\Omega)\leq\frac{1}{2}$;

(iv) $M \cdot mes\,(\Omega) \leq r$, *where* M *is a positiv constant for which*

$$||K(t,s,u_1,u_2)||_{L^2(\Omega)} \leq M,$$

for any $t,s \in \Omega$ *and* $u_1, u_2 \in \mathbb{R}$.
Then the equation (2) *has a solution in* $\overline{B}\,(f,r) \subset L^2(\Omega)$.

Denote

$$\left(\widetilde{A}x\right)(t) = \int\limits_{\Omega} K(t,s,x(s),x(g(s)))ds,$$

$$(Ax)\,(t) = f(t) + \left(\widetilde{A}x\right)(t).$$

The set of the solutions of the integral equation (2) coincides with the set of fixed points of the operator A in $L^2(\Omega)$.
We show that $A : \overline{B}\,(f,r) \to \overline{B}\,(f,r)$. We take $x \in \overline{B}\,(f,r)$ and we prove that $Ax \in \overline{B}\,(f,r)$.
For $x \in \overline{B}\,(f,r) \Leftrightarrow ||x-f||_{L^2(\Omega)} \leq r$
$(Ax)\,(t) - f(t) = \left(\widetilde{A}x\right)(t) = \int\limits_{\Omega} K(t,s,x(s),x(g(s)))ds.$

Using the norm we obtain that $||Ax-f||_{L^2(\Omega)} \leq M \cdot mes(\Omega) \leq r$, so $Ax \in \overline{B}(f,r)$, A is well defined and

$$A\left(\overline{B}\,(f,r)\right) \subset \overline{B}\,(f,r)\,.$$

We show that A is a nonexpansive operator.

$$|(Ax)(t)-(Ay)(t)| = \left|\int\limits_{\Omega} (K(t,s,x(s),x(g(s))) - K(t,s,y(s),y(g(s)))ds)\right| \leq$$

$$\int\limits_{\Omega} |K(t,s,x(s),x(g(s))) - K(t,s,y(s),y(g(s)))|\,ds \leq \int\limits_{\Omega} L(t,s)\cdot(|x(s)-y(s)| +$$

$$+ |x(g(s)) - y(g(s))|)ds$$

Using the $L^2(\Omega)$ norm in last inequality, we obtain:

$$||Ax-Ay||_{L^2(\Omega)} \leq ||L||_{L^2(\Omega\times\Omega)} \cdot 2 \cdot ||x-y||_{L^2(\Omega)} \cdot mes(\Omega)$$

For $||L||_{L^2(\Omega)} \cdot mes(\Omega) \leq \frac{1}{2}$ we obtain:

$$||Ax-Ay||_{L^2(\Omega)} \leq ||x-y||_{L^2(\Omega)}.$$

In the case $||L||_{L^2(\Omega)} \cdot mes(\Omega) < \frac{1}{2}$ the result follows by Theorem 1.2 using the technique of Picard operators. In equality case A is a nonexpansive mapping and using Browder-Ghode-Kirk theorem (Theorem 1.1) for $A : \overline{B}(f,r) \to \overline{B}(f,r)$,it results that A has a fixed point. We know that $\overline{B}(f,r)$ is nonempty, bounded, convex and closed subset of $L^2(\Omega)$.
The integral equation (2) has a solution in $B(f,r) \subset L^2(\Omega)$ but is not unique.

Remark 1. If $\Omega = [a,b]$ then by Theorem 1.2 we obtain a result for equation (2), see [3].

Generalization

We consider the integral equation with modified argument :

$$x(t) = f(t) + \int_{\Omega} K(t,s,x(g_1(s)),...,x(g_m(s)))ds \tag{3}$$

where:
$K : \overline{\Omega} \times \overline{\Omega} \times \mathbb{R}^m \to \mathbb{R}$;
$f : \overline{\Omega} \to \mathbb{R}$;
$g_k : \overline{\Omega} \to \overline{\Omega},\ k = \overline{1,m}$.

Theorem 4. *Assume that the following conditions are satisfied:*

(i) $K(t,s,x(g_1),...,x(g_m))$ *satisfied the Caratheodory condition, i.e*

(a) $K(t,s,\bullet,\bullet),...,\bullet)$ *is continuous almost everywhere in* $\mathbb{R}^m$ *for almost all* $t \in \Omega$ *and* $s \in \Omega$*;*

(b) $K(\bullet,\bullet,x(g_1),...,x(g_m))$ *is measurable in* $\overline{\Omega} \times \overline{\Omega}$*, for any* $x(g_i) \in \mathbb{R}^m,\ i = \overline{1,m}$ *fixed.*

(ii) $f \in L^2(\Omega)$ *and there exists* $L \in L^2(\Omega \times \Omega)$ *such that*

$$|K(t,s,u_1,u_2,...,u_m) - K(t,s,v_1,v_2,...,v_m)| \leq L(t,s) \cdot \sum_{i=1}^{m} \|u_i - v_i\|_{L^2(\Omega)},$$

for any $t,s \in \Omega$ *and* $u_i, v_i \in \mathbb{R}^m, i = \overline{1,m}$.

(iii) $||L||_{L^2(\Omega\times\Omega)} \cdot mess(\Omega) \leq \frac{1}{m}$*;*

(iv) $M \cdot mes(\Omega) \leq r$*, where* M *is a positiv constant for which*

$$||K(t,s,u_1,u_2,...,u_m)||_{L^2(\Omega)} \leq M,$$

for any $t,s \in \Omega$ *and* $u_i \in \mathbb{R},\ i = \overline{1,m}$.
Then the equation (3) *has a solution in* $\overline{B}(f,r) \subset L^2(\Omega)$.

Similarly, we define the operator

$$(Ax)(t) = f(t) + \int_{\Omega} K(t,s,x(g_1(s)),...,x(g_m(s)))ds, \;\; x \in \overline{B}(f,r).$$

In similarly case the operator A map $\overline{B}(f,r) \subset L^2(\Omega)$ in itself. We obtain

$$\|Ax - Ay\|_{L^2(\Omega)} \leq \|L\|_{L^2(\Omega\times\Omega)} \cdot m \cdot mess(\Omega) \cdot \|x-y\|$$

Thus, by the same proof as given in Theorem 2.3, we can prove that in equality case A is nonexpansive map and it has a fixed point.The solution of integral equation (3) is the fixed point of operator A.

3. Examples

Example 1. Consider the integral equation with deviating argument:

$$x(t) = t^2 + \int_1^a e^{-t} \cdot s^2 \cdot x(s) \cdot x(s - e^{-s})ds \tag{4}$$

Note that the above equation represent a special case of (2.2) where

$$f(t) = t^2, \quad K(t,s,x(s),x(g(s))) = e^{-t} \cdot s^2 \cdot x(s) \cdot x(s - e^{-s}).$$

$$\overline{B}(t^2,r) = \left\{x \in L^2(\Omega)/ \left|\left|x - t^2\right|\right|_{L^2(\Omega)} \leq r\right\},$$

where $\Omega = [1,a]$. It is easily seen that for (4), the assumption (ii) of Theorem 2.3 is satisfied with $L(t,s) = e^{-t} \cdot s^2$.

In this case the assumption (iv) is satisfied for the constant $M = a^2 \cdot r^2$ and $r \leq \frac{1}{a^2(a-1)}$. Moreover the assumption (iii) of Theorem 2.3 is satisfied for

$$(a-1) \cdot \sqrt{\int_1^a \int_1^a (e^{-t} \cdot s^2)^2 \cdot dsdt} = \sqrt{\frac{-e^{-2a} + e^{-2}}{2} \cdot \frac{a^5 - 1}{5}} \cdot (a-1) \leq \frac{1}{2}$$

In this case we obtain $a \approx 1.29$. Note that Theorem 1.2 cannot be applied.

For a good approximation of M we take $M = \frac{a^2}{e}$ and we obtain from (iii) and (iv)

$$r \leq \frac{e}{a^2 \cdot (a-1)} \; and \; a^2(a-1) \leq \frac{e}{2}$$

with a solution of inequation $a \approx 1.559$

So, for $\Omega = [1;a]$ the equation (4) has a solution in $\overline{B}(t^2,r) \subset L^2(\Omega)$.

Example 2. Consider the integral equation with deviating argument:

$$x(t) = t + \int_{1}^{a} \left[e^{-t} \cdot x(s) + s \cdot x(s - e^{-s})\right] ds \tag{5}$$

In this case $f(t) = t$, $K(t, s, x(s), x(g(s))) = e^{-t} \cdot x(s) + s \cdot x(s - e^{-s})$ and

$$\overline{B}(t, r) = \left\{x \in L^2(\Omega)/ \,||x - t|\,|_{L^2(\Omega)} \leq r\right\},$$

where $\Omega = [1, a]$.
The assumption (ii) of Theorem 2.3 is satisfied for $L(t, s) = \max\{e^{-t}, s\}$ for $t, s \in [1, a]$. From the assumptions (iii) and (iv) and a good approximation of $L(t, s)$ and $M = a^2(r + 1)^2$, we obtain the inequation

$$(a - 1)^2 \cdot \sqrt{\frac{a^2 + a + 1}{3}} \leq (a - 1)^2 \cdot a \leq \frac{1}{2}$$

with a solution of inequation $a = 1,565$ and $a^2 \cdot (r + 1)^2 \cdot (a - 1) \leq r$. In this case, for equality, the Theorem 1.2 cannot be applied.
The equation (5) has a solution in $\overline{B}(t, r) \subset L^2(\Omega)$.

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