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TURN-TYPE INEQUALITIES FOR THE GAMMA AND POLYGAMMA FUNCTIONS

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Abstract. The aim of this paper is to establish new Turán-type inequalities involving the polygamma functions, which are stronger than the inequalities established by A. Laforgia and P. Natalini [J. Inequal. Pure Appl. Math., 27 (2006), Issue 1, Art. 32].

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1.Introduction

The inequalities of the type

$$f_n(x)\, f_{n+2}(x) - f_{n+1}^2(x) \leq 0$$

have many applications in pure mathematics as in other branches of science. They are named by Karlin and Szegö [3], Turán-type inequalities because the first of these type of inequalities was introduced by Turán [6]. More precisely, he used some results of Szegö [5] to prove the previous inequality for $x \in (-1, 1)$, where f_n is the Legendre polynomial of degree n. This classical result has been extended in many directions, as ultraspherical polynomials, Lagguere and Hermite polynomials, or Bessel functions, and so forth. There is today a huge literature on Turán inequalities, since they have important applications in complex analysis, number theory, combinatorics, theory of mean-values, or statistics and control theory.

Recently, Laforgia and Natalini [4] proved some Turán-type inequalities for some special functions as well as the polygamma functions, by using the following inequality:

$$\int_a^b g(t)\, f^m(t)\, dt \cdot \int_a^b g(t)\, f^n(t)\, dt \geq \left(\int_a^b g(t)\, f^{\frac{m+n}{2}}(t)\, dt \right)^2, \tag{1.1}$$

where f, g are non-negative functions such that these integrals exist.

This inequality is considered in [4] as a generalization of the Cauchy-Schwarz inequality, but it can be also viewed as a particular case of the Cauchy-Schwarz inequality, for $t \mapsto (g(t)\, f^m(t))^{1/2}$ and $t \mapsto (g(t)\, f^n(t))^{1/2}$.

2. Applying Hölder inequality

We use here the Hölder inequality

$$\left(\int_a^b u^p(t)\,dt\right)^{1/p}\left(\int_a^b v^q(t)\,dt\right)^{1/q}\geq\int_a^b u(t)\,v(t)\,dt,$$

where $p,q>0$ are such that $p^{-1}+q^{-1}=1$ and u, v are non-negative functions such that these integrals exist. Case $p=q=2$ is the Cauchy-Schwarz inequality.

By taking $u(x)=g(t)^{1/p}f^{m/p}(t)$ and $v(x)=g(t)^{1/q}f^{n/q}(t)$, we can state the following extension of the inequality (1.1):

$$\left(\int_a^b g(t)\,f^m(t)\,dt\right)^{1/p}\left(\int_a^b g(t)\,f^n(t)\,dt\right)^{1/q}\geq\int_a^b g(t)\,f^{\frac{m}{p}+\frac{n}{q}}(t)\,dt. \tag{2.1}$$

In what follows, we use the integral representations, for $x>0$ and $n=1,2,...$

$$\Gamma^{(n)}(x)=\int_0^\infty e^{-t}t^{x-1}\log^n t dt, \tag{2.2}$$

$$\psi^{(n)}(x)=(-1)^{n+1}\int_0^\infty\frac{t^n e^{-tx}}{1-e^{-t}}dt, \tag{2.3}$$

and

$$\zeta(x)=\frac{1}{\Gamma(x)}\int_0^\infty\frac{t^{x-1}}{e^t-1}dt,\quad x>1, \tag{2.4}$$

where Γ is the gamma function, $\psi^{(n)}$ is the n-th polygamma function and ζ is the Riemann-zeta function. See, for instance, [1, p. 260].

In this section, we first give an extension of the main result of Laforgia and Natalini [4, Theorem 2.1].

Theorem 2.1. *For every p, $q>0$ with $p^{-1}+q^{-1}=1$ and for every integers m, $n\geq 1$ such that $\frac{m}{p}+\frac{n}{q}$ is an integer, we have:*

$$\left|\psi^{(m)}(x)\right|^{1/p}\cdot\left|\psi^{(n)}(x)\right|^{1/q}\geq\left|\psi^{\left(\frac{m}{p}+\frac{n}{q}\right)}(x)\right|.$$

Proof. We choose $g(t)=\frac{e^{-tx}}{1-e^{-t}}$, $f(t)=t$, and $a=0$, $b=+\infty$ in (2.1) to get

$$\left(\int_0^\infty\frac{t^m e^{-tx}}{1-e^{-t}}dt\right)^{1/p}\left(\int_0^\infty\frac{t^n e^{-tx}}{1-e^{-t}}dt\right)^{1/q}\geq\int_0^\infty\frac{t^{\frac{m}{p}+\frac{n}{q}}e^{-tx}}{1-e^{-t}}dt$$

and the conclusion follows using (2.3).□

The next result extends Theorem 2.2 from Laforgia and Natalini [4].

Theorem 2.2. *For every* x, y, p, $q > 0$ *such that* $p^{-1} + q^{-1} = 1$, *we have:*

$$\zeta^{1/p}(x)\,\zeta^{1/q}(y) \geq \frac{\Gamma\left(\frac{x}{p}+\frac{y}{q}\right)}{\Gamma^{1/p}(x)\,\Gamma^{1/q}(y)}\zeta\left(\frac{x}{p}+\frac{y}{q}\right).\square$$

Proof. We choose $g(t) = \frac{1}{1-e^{-t}}$, $f(t) = t$, and $a = 0$, $b = +\infty$ in (2.1) to get

$$\left(\int_0^\infty \frac{t^{x-1}}{e^t-1}dt\right)^{1/p}\left(\int_0^\infty \frac{t^{y-1}}{e^t-1}dt\right)^{1/q} \geq \int_0^\infty \frac{t^{\frac{x-1}{p}+\frac{y-1}{q}}}{e^t-1}dt,$$

or

$$(\Gamma(x)\,\zeta(x))^{1/p}\,(\Gamma(y)\,\zeta(y))^{1/q} \geq \Gamma\left(\frac{x}{p}+\frac{y}{q}\right)\zeta\left(\frac{x}{p}+\frac{y}{q}\right),$$

which is the conclusion.$\square$

Particular case $p = q = 2$, $x = s$, $y = s+2$ is the object of Theorem 2.3 from [4].

3.Turán type inequalities for $\exp\Gamma^{(n)}(x)$ and $\exp\psi^{(n)}(x)$

Very recently, Alzer and Felder [2] proved the following sharp inequality for Euler's gamma function,

$$\alpha \leq \Gamma^{(n-1)}(x)\,\Gamma^{(n+1)}(x) - \left(\Gamma^{(n)}(x)\right)^2,$$

for odd $n \geq 1$, and $x > 0$, where $\alpha = \min_{1.5\leq x\leq 2}\left(\psi'(x)\,\Gamma^2(x)\right) = 0.6359...$.

We prove similar results about the sequences $\exp\Gamma^{(n)}(x)$, and $\exp\psi^{(n)}(x)$.

Theorem 3.1. *For every* $x > 0$ *and even integers* $n \geq k \geq 0$, *we have*

$$\left(\exp\Gamma^{(n)}(x)\right)^2 \leq \exp\Gamma^{(n+k)}(x)\cdot\exp\Gamma^{(n-k)}(x).$$

Proof. We use (2.2) to estimate the expression

$$\frac{\Gamma^{(n-k)}(x)+\Gamma^{(n+k)}(x)}{2} - \Gamma^{(n)}(x) =$$

$$= \frac{1}{2}\left(\int_0^\infty e^{-t}t^{x-1}\log^{n-k}t\,dt + \int_0^\infty e^{-t}t^{x-1}\log^{n+k}t\,dt\right) - \int_0^\infty e^{-t}t^{x-1}\log^n t\,dt =$$

$$= \frac{1}{2}\int_0^\infty\left(\frac{1}{\log^k t}+\log^k t - 2\right)e^{-t}t^{x-1}\log^n t\,dt \geq 0.$$

The conclusion follows by exponentiating the inequality

$$\frac{\Gamma^{(n-k)}(x)+\Gamma^{(n+k)}(x)}{2}\geq\Gamma^{(n)}(x).\square$$

Theorem 3.2. *For every $x>0$ and integers $n\geq 1$, we have:*
(i) *If n is odd, then* $\left(\exp\psi^{(n)}(x)\right)^2\geq\exp\psi^{(n+1)}(x)\cdot\exp\psi^{(n-1)}(x)$.
(ii) *If n is even, then* $\left(\exp\psi^{(n)}(x)\right)^2\leq\exp\psi^{(n+1)}(x)\cdot\exp\psi^{(n-1)}(x)$.
Proof. We use (2.3) to estimate the expression

$$\psi^{(n)}(x)-\frac{\psi^{(n+1)}(x)+\psi^{(n-1)}(x)}{2}=$$

$$=(-1)^{n+1}\left(\int_0^\infty\frac{t^ne^{-tx}}{1-e^{-t}}dt+\frac{1}{2}\int_0^\infty\frac{t^{n+1}e^{-tx}}{1-e^{-t}}dt+\frac{1}{2}\int_0^\infty\frac{t^{n-1}e^{-tx}}{1-e^{-t}}dt\right)=$$

$$=\frac{(-1)^{n+1}}{2}\int_0^\infty\frac{t^{n-1}e^{-tx}}{1-e^{-t}}(t+1)^2\,dt.$$

Now, the conclusion follows by exponentiating the inequality

$$\psi^{(n)}(x)\geq(\leq)\,\frac{\psi^{(n+1)}(x)+\psi^{(n-1)}(x)}{2},$$

as n is odd, respective even.$\square$

4. Concluding remarks

It is mentioned in the final part of the paper [4] that many other Turán-type inequalities can be obtained for the functions which admit integral representations of the type (2.2)-(2.4). As an example, for the exponential integral function [1, p. 228, Rel. 5.1.4]

$$E_n(x)=\int_0^\infty e^{-tx}t^ndt\ ,\qquad x>0,\quad n=0,1,2,...,$$

and using the inequality (1.1), we deduce that for $x>0$ and positive integers m, n such that $\frac{m+n}{2}$ is an integer,

$$E_n(x)\,E_m(x)\geq E_{\frac{n+m}{2}}(x).\tag{4.1}$$

Using again (2.1), we can establish the following extension:

Theorem 4.1. *For every p, q, $x > 0$ with $p^{-1} + q^{-1} = 1$ and for every integers m, $n \geq 1$ such that $\frac{m}{p} + \frac{n}{q}$ is an integer, it holds*

$$(E_m(x))^{1/p} (E_n(x))^{1/q} \geq E_{\frac{m}{p}+\frac{n}{q}}(x).$$

This follows from (2.1), with $g(t) = e^{-tx}$, $f(t) = t$, $a = 0$, $b = +\infty$. The particular case (4.1) is obtained for $p = q = 2$.

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