Acta Universitatis Apulensis No. 23/2010
ISSN: 1582-5329 pp. 107-115

SOME INCLUSION PROPERTIES FOR NEW SUBCLASSES OF MEROMORPHIC P-VALENT STRONGLY STARLIKE AND STRONGLY CONVEX FUNCTIONS ASSOCIATED WITH THE EL-ASHWAH OPERATOR

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Abstract. The purpose of this paper is to derive some useful properties for new subclasses of strongly starlike and strongly convex functions of order γ and type (μ_1, μ_2) in the open unit disk $\mathcal{U}$ using a multiplier transformation for meromorphic p-valent functions introduced recently by R.M. El-Ashwah. Inclusion relationships using these subclasses are established.

2010 *Mathematics Subject Classification*: Primery: 30C45; Secondary: 30C55.

Keywords and phrases: Analytic functions, Meromorphic p-valent functions, Multiplier transformations, Strongly starlike functions, Strongly convex functions.

1. Introduction and definitions

Let Σ_p denote the class of functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} a_n z^n \qquad (p \in N = \{1, 2, ...\}), \tag{1.1}$$

which are analytic in the punctured unit disk $\mathcal{U}^* = \{z : z \in \mathbb{C}; 0 < |z| < 1\} = \mathcal{U}\backslash\{0\}$, where $\mathcal{U} = \{z : z \in \mathbb{C}; |z| < 1\}$ is the open unit disk.

The classes of strongly starlike functions and strongly convex functions in the open unit disk have earlier been introduced and studied by Takahashi and Nunokawa [9], Shanmugam et al. [7] and others. A function $f \in \Sigma_p$ is said to belong to the class of meromorphically strongly starlike functions of order γ and type (μ_1, μ_2) in $\mathcal{U}$, denoted by $\mathcal{S}_p^*(\mu_1, \mu_2, \gamma)$, if it satisfies

$$-\frac{\pi}{2}\,\mu_1 < \arg\left\{\frac{z\,f'(z)}{f(z)} + \gamma\right\} < \frac{\pi}{2}\,\mu_2 \tag{1.2}$$

$$(z \in \mathcal{U},\ 0 < \mu_1 \leq 1,\ 0 < \mu_2 \leq 1,\ \gamma > p).$$

A function $f \in \Sigma_p$ is said to belong to the class of meromorphically strongly convex functions of order γ and type (μ_1, μ_2) in $\mathcal{U}$, denoted by $\mathcal{C}_p(\mu_1, \mu_2, \gamma)$, if it satisfies

$$-\frac{\pi}{2}\,\mu_1 < \arg\left\{\frac{(z\,f'(z))'}{f'(z)} + \gamma\right\} < \frac{\pi}{2}\,\mu_2 \tag{1.3}$$

$$(z \in \mathcal{U},\ 0 < \mu_1 \leq 1,\ 0 < \mu_2 \leq 1,\ \gamma > p).$$

Recently El-Ashwah [4] defined the operator

$$I_p^m(\lambda, \ell)f(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty}\left(\frac{\ell + \lambda(n+p)}{\ell}\right)^m a_n\,z^n \tag{1.4}$$

$$(\lambda \geq 0;\ \ell > 0;\ m \in N_0 = N \cup \{0\};\ z \in \mathcal{U}^*).$$

It is easily verified from (1.4) that

$$\lambda z(I_p^m(\lambda, \ell)\,f(z))' \;=\; \ell\,I_p^{m+1}(\lambda, \ell)\,f(z) - (\ell + \lambda p)\,I_p^m(\lambda, \ell)\,f(z) \qquad (\lambda > 0). \tag{1.5}$$

We note that

$$I_p^0(\lambda, \ell)\,f(z) = f(z) \quad \text{and}$$

$$I_p^1(1,1)\,f(z) \;=\; \frac{(z^{p+1}\,f(z))'}{z^p} = (p+1)f(z) + z\,f'(z).$$

Also by specializing the parameters λ, l and p, we obtain the following operators studied earlier by various authors:

(i) $I_1^m(1, \ell)f(z) = I(m, \ell)f(z)$ (see Cho et al. [2,3]);

(ii) $I_p^m(1,1)f(z) = D_p^m f(z)$ (see Aouf and Hossen [1], Liu and Owa [5] and Srivastava and Patel [8]);

(iii) $I_1^m(1,1)f(z) = I^m f(z)$ (see Uralegaddi and Somanatha [10]).

Also we note that:

(i) $I_p^m(1, \ell)f(z) = I_p(m, \ell)f(z)$, where $I_p(m, \ell)f(z)$ is defined by

$$I_p(m, \ell)f(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty}\left(\frac{\ell + n + p}{\ell}\right)^m a_n\,z^n \quad (\ell > 0;\ m \in N_0;\ z \in \mathcal{U}^*); \tag{1.6}$$

(ii) $I_p^m(\lambda,1)f(z) = D_{\lambda,p}^m f(z)$, where $D_{\lambda,p}^m f(z)$ is defined by

$$D_{\lambda,p}^m f(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} [\lambda(n+p)+1]^m \, a_n \, z^n \quad (\lambda \geq 0;\, m \in N_0;\, z \in \mathcal{U}^*). \tag{1.7}$$

Now, we introduce the following new subclasses of meromorphically strongly starlike and meromorphically strongly convex functions of order γ and type (μ_1,μ_2) in the open unit disk $\mathcal{U}$:

$$R_p^m(\lambda,\ell;\mu_1,\mu_2,\gamma) = \left\{ f \in \Sigma_p : I_p^m(\lambda,\ell)f \in \mathcal{S}_p^*(\mu_1,\mu_2,\gamma),\, -\frac{z\,(I_p^m(\lambda,\ell)f(z))'}{I_p^m(\lambda,\ell)f(z)} \neq \gamma,\, z \in \mathcal{U} \right\} \tag{1.8}$$

and

$$M_p^m(\lambda,\ell;\mu_1,\mu_2,\gamma) = \left\{ f \in \Sigma_p : I_p^m(\lambda,\ell)f \in \mathcal{C}_p(\mu_1,\mu_2,\gamma),\, -\frac{[z((I_p^m(\lambda,\ell)f)'(z))]'}{[I_p^m(\lambda,\ell)f(z)]'} \neq \gamma,\, z \in \mathcal{U} \right\}. \tag{1.9}$$

The object of this paper is to derive some properties for the classes $R_p^m(\lambda,\ell;\mu_1,\mu_2,\gamma)$ and $M_p^m(\lambda,\ell;\mu_1,\mu_2,\gamma)$.

2. Main results

In order to prove our results, we need the following result of Nunokawa et al. [6]:
Lemma 2.1. *Let $q(z)$ be analytic in $\mathcal{U}$ with $q(0)=1$ and $q(z)\neq 0$. If there exists two points $z_1,z_2 \in \mathcal{U}$ such that*

$$-\frac{\pi}{2}\,\mu_1 = \arg q(z_1) < \arg q(z) < \arg q(z_2) = \frac{\pi}{2}\mu_2\,, \tag{2.1}$$

for $\mu_1>0$, $\mu_2>0$, and for $|z|<|z_1|=|z_2|$, then we have

$$\frac{z_1\, q'(z_1)}{q(z_1)} = -i\,\frac{\mu_1+\mu_2}{2}\,k, \tag{2.2}$$

and

$$\frac{z_2\, q'(z_2)}{q(z_2)} = i\,\frac{\mu_1+\mu_2}{2}\,k, \tag{2.3}$$

where $k \geq \frac{1-|a|}{1+|a|}$ and $a = i\tan\left\{\frac{\pi}{4}\left(\frac{\mu_2-\mu_1}{\mu_2+\mu_1}\right)\right\}$. (2.4)

Theorem 2.2.

$$R_p^{m+1}(\lambda,\ell;\mu_1,\mu_2,\gamma) \subset R_p^m(\lambda,\ell;\mu_1,\mu_2,\gamma), \textit{ for each } m \in N_0.$$

Proof. Let $f \in R_p^{m+1}(\lambda,\ell;\mu_1,\mu_2,\gamma)$. Further suppose that

$$\frac{z\,(I_p^m(\lambda,\ell)f(z))'}{I_p^m(\lambda,\ell)f(z)} = (\gamma-p)\,q(z)-\gamma\,, \tag{2.5}$$

where $q(z) = 1 + c_1 z + c_2 z^2 + ...$ is analytic in $\mathcal{U}$, $q(0) = 1$, and $q(z) \neq 0\ \forall\, z \in \mathcal{U}$. Using (1.5), we get

$$\frac{\ell\, I_p^{m+1}(\lambda,\ell)f(z)}{I_p^m(\lambda,\ell)f(z)} = \lambda(\gamma-p)\,q(z) + [\ell - \lambda(\gamma-p)]\,. \tag{2.6}$$

By logarithmic differentiation, we easily get

$$\frac{z\,(I_p^{m+1}(\lambda,\ell)f(z))'}{I_p^{m+1}(\lambda,\ell)f(z)} + \gamma = \frac{z\,(I_p^m(\lambda,\ell)f(z))'}{I_p^m(\lambda,\ell)f(z)} + \frac{\lambda(\gamma-p)\,z\,q'(z)}{\lambda(\gamma-p)\,q(z)+[\ell-\lambda(\gamma-p)]} + \gamma$$

$$= (\gamma-p)\,q(z) + \frac{\lambda(\gamma-p)\,z\,q'(z)}{\lambda(\gamma-p)\,q(z)+[\ell-\lambda(\gamma-p)]}. \tag{2.7}$$

Suppose that there exist two points $z_1, z_2 \in \mathcal{U}$ such that,

$$-\tfrac{\pi}{2}\,\mu_1 = \arg q(z_1) < \arg q(z) < \arg q(z_2) = \tfrac{\pi}{2}\mu_2 \quad \text{for } |z| < |z_1| = |z_2|\,.$$

Then from the proof of the Nunokawa lemma [6], we have

$$\frac{z_1 q'(z_1)}{q(z_1)} = -\frac{i\,k(\mu_1+\mu_2)(1+t_1^2)}{4\,t_1} \tag{2.8}$$

and

$$\frac{z_2 q'(z_2)}{q(z_2)} = \frac{i\,k(\mu_1+\mu_2)(1+t_2^2)}{4\,t_2}, \tag{2.9}$$

where

$$q(z_1) = (-it_1)^{(\mu_1+\mu_2)/2}\ \exp\left\{i\frac{\pi}{4}(\mu_2-\mu_1)\right\}, \quad t_1 > 0 \tag{2.10}$$

$$q(z_2) = (it_2)^{(\mu_1+\mu_2)/2}\ \exp\left\{i\frac{\pi}{4}(\mu_2-\mu_1)\right\}, \quad t_2 > 0 \tag{2.11}$$

and

$$k \geq \frac{1-|a|}{1+|a|}.$$

Replacing z by z_2 in (2.7) and using (2.9) and (2.11) therein, we find that

$$\frac{z_2(I_p^{m+1}(\lambda,\ell)f(z_2))'}{I_p^{m+1}(\lambda,\ell)f(z_2)} + \gamma$$

$$= (\gamma - p)\, q(z_2) \, \left[1 + \frac{\lambda z_2\, q'(z_2)/q(z_2)}{\lambda(\gamma - p)q(z_2) + [\ell - (\gamma - p)]}\right]$$

$$= (\gamma{-}p)\, t_2^{(\mu_1+\mu_2)/2} \exp\left(i\frac{\pi}{2}\mu_2\right) \left[1 + \frac{\lambda i(\mu_1 + \mu_2)\,(1 + t_2^2)k}{4t_2[\lambda(\gamma - p)t_2^{(\mu_1+\mu_2)/2} \, \exp\left(i\frac{\pi}{2}\mu_2\right) + \{\ell - \lambda(\gamma - p)\}]}\right].$$

This implies that

$$\arg\left\{\frac{z_2(I_p^{m+1}(\lambda,\ell)f(z_2))'}{I_p^{m+1}(\lambda,\ell)f(z_2)} + \gamma\right\}$$

$$= \frac{\pi}{2}\,\mu_2 + \arg\left\{1 + \frac{\lambda i(\mu_1 + \mu_2)\,(t_2^{-1} + t_2)k}{4\,[\lambda(\gamma - p)t_2^{(\mu_1+\mu_2)/2} \exp\left(i\frac{\pi}{2}\mu_2\right) + \{\ell - \lambda(\gamma - p)\}]}\right\}$$

$$= \frac{\pi}{2}\mu_2{+}\tan^{-1}\left\{\frac{k(\mu_1 + \mu_2)\,(t_2^{-1} + t_2)[\ell - \lambda(\gamma - p)] + 4\lambda(\gamma - p)\, t_2^{(\mu_1+\mu_2)/2} \cos\frac{\pi}{2}\mu_2}{4\,\varepsilon(\mu_1,\mu_2,t_2)}\right\}$$

$$\geq \tfrac{\pi}{2}\mu_2\,, \tag{2.12}$$

where

$$\varepsilon(\mu_1,\mu_2,t_2) = [\ell - \lambda(\gamma - p)]^2 + 2\lambda\,[\ell - \lambda(\gamma - p)]\,(\gamma - p)\, t_2^{(\mu_1+\mu_2)/2} \cos\tfrac{\pi}{2}\mu_2$$
$$+\lambda^2(\gamma{-}p)^2 t_2^{(\mu_1+\mu_2)} {+} k\lambda^2\, \left(\tfrac{\mu_1+\mu_2}{4}\right) (t_2^{-1}{+}t_2)\,(\gamma{-}p) t_2^{(\mu_1+\mu_2)/2} \sin\tfrac{\pi}{2}\mu_2\,,$$

and

$$k \geq \frac{1 - |a|}{1 + |a|}.$$

Similarly, replacing $z = z_1$ in (2.8) and using the same procedure as above, we obtain that

$$\arg\left\{\frac{z_1(I_p^{m+1}(\lambda,\ell)f(z_1))'}{I_p^{m+1}(\lambda,\ell)f(z_1)} + \gamma\right\} \leq -\frac{\pi}{2}\,\mu_1. \tag{2.13}$$

Thus we get the contradiction to the hypothesis that $f \in R_p^{m+1}(\lambda,\ell;\mu_1,\mu_2,\gamma)$. Hence the function $q(z)$ defined by (2.5) yields

$$-\frac{\pi}{2}\mu_1 < \arg q(z) < \frac{\pi}{2}\,\mu_2\,,$$

which implies that

$$-\frac{\pi}{2}\mu_1 < \arg\left\{\frac{z(I_p^m(\lambda,\ell)f(z))'}{I_p^m(\lambda,\ell)f(z)} + \gamma\right\} < \frac{\pi}{2}\,\mu_2\,.$$

Thus $f \in R_p^m(\lambda,\ell;\mu_1,\mu_2,\gamma)$, which completes the proof of the Theorem 2.2.

On taking $\lambda = 1$ in Theorem 2.2 and using (1.6), we get the following result:

Corollary 2.3.

$$R_p^{m+1}(\ell;\mu_1,\mu_2,\gamma)\subset R_p^m(\ell;\mu_1,\mu_2,\gamma),\ \textit{for each } m\in N_0$$

where

$$R_p^m(\ell;\mu_1,\mu_2,\gamma)=\left\{f\in\Sigma_p: I_p(m,\ell)f\in\mathcal{S}_p^*(\mu_1,\mu_2,\gamma),\ -\frac{z(I_p(m,\ell)f(z))'}{I_p(m,\ell)f(z)}\neq\gamma,\ z\in\mathcal{U}\right\}, \tag{2.14}$$

and $I_p(m,\ell)f(z)$ is given by (1.6).

If we put $\ell=1$ in Theorem 2.2 and use (1.7) therein, we get

Corollary 2.4.

$$R_p^{m+1}(\lambda;\mu_1,\mu_2,\gamma)\subset R_p^m(\lambda;\mu_1,\mu_2,\gamma),\ \textit{for each } m\in N_0$$

where

$$R_p^m(\lambda;\mu_1,\mu_2,\gamma)=\left\{f\in\Sigma_p: D_{\lambda,p}^m f\in\mathcal{S}_p^*(\mu_1,\mu_2,\gamma),\ -\frac{z(D_{\lambda,p}^m f(z))'}{D_{\lambda,p}^m f(z)}\neq\gamma,\ z\in\mathcal{U}\right\}, \tag{2.15}$$

and $D_{\lambda,p}^m f(z)$ is given by (1.7).

Theorem 2.5.

$$M_p^{m+1}(\lambda,\ell;\mu_1,\mu_2,\gamma)\subset M_p^m(\lambda,\ell;\mu_1,\mu_2,\gamma),\ \textit{for each } m\in N_0.$$

Proof.

$$\begin{aligned} f\in M_p^{m+1}(\lambda,\ell;\mu_1,\mu_2,\gamma) &\iff I_p^{m+1}(\lambda,\ell)f(z)\in\mathcal{C}_p(\mu_1,\mu_2,\gamma)\\ &\iff -\tfrac{1}{p}z(I_p^{m+1}(\lambda,\ell)f(z))'\in\mathcal{S}_p^*(\mu_1,\mu_2,\gamma)\\ &\iff I_p^{m+1}(\lambda,\ell)\left(-\tfrac{1}{p}zf'(z)\right)\in\mathcal{S}_p^*(\mu_1,\mu_2,\gamma)\\ &\iff -\tfrac{1}{p}zf'(z)\in R_p^{m+1}(\lambda,\ell;\mu_1,\mu_2,\gamma)\\ &\iff -\tfrac{1}{p}zf'(z)\in R_p^{m}(\lambda,\ell;\mu_1,\mu_2,\gamma)\\ &\iff I_p^{m}(\lambda,\ell)\left(-\tfrac{1}{p}zf'(z)\right)\in\mathcal{S}_p^*(\mu_1,\mu_2,\gamma)\\ &\iff -\tfrac{1}{p}z\,(I_p^{m}(\lambda,\ell)f(z))'\in\mathcal{S}_p^*(\mu_1,\mu_2,\gamma)\\ &\iff I_p^{m}(\lambda,\ell)f(z)\in\mathcal{C}_p(\mu_1,\mu_2,\gamma)\\ &\iff f\in M_p^m(\lambda,\ell;\mu_1,\mu_2,\gamma). \end{aligned}$$

For $\lambda=1$ and $\ell=1$ in Theorem 2.5, we get results similar to Corollary 2.3 and 2.4 respectively.

It is obvious from Theorem 2.5 that Alexander's type relationship holds between the classes $M_p^m(\lambda;\mu_1,\mu_2,\gamma)$ and $R_p^m(\lambda;\mu_1,\mu_1,\gamma)$, which we state formally as:

Theorem 2.6.

$$f\in M_p^m(\lambda,\ell;\mu_1,\mu_2,\gamma)\iff -\tfrac{1}{p}zf'\in R_p^m(\lambda,\ell;\mu_1,\mu_2,\gamma).$$

3. Integral operators

Let $f \in \Sigma_p$. Then for $\mu > 0$, we consider the integral operator $F_{\mu,p}(f)(z) : \Sigma_p \to \Sigma_p$ defined by

$$F_{\mu,p}(f)(z) = \frac{\mu}{z^{\mu+p}} \int_0^z t^{\mu+p-1} f(t)\, dt$$

$$= \left[\frac{1}{z^p} + \sum_{n=0}^{\infty} \left(\frac{\mu}{n+\mu+p} \right) z^n \right] * f(z). \qquad (3.1)$$

Theorem 3.1. *If $f \in R_p^m(\lambda, \ell; \mu_1, \mu_2, \gamma)$, and for $\mu > 0$,*

$$- \frac{z\,[I_p^m(\lambda,\ell)(F_{\mu,p}(f)(z))]'}{[I_p^m(\lambda,\ell)(F_{\mu,p}(f)(z))]} \neq \gamma \quad \forall z \in \mathcal{U},$$

then $F_{\mu,p}(f)(z) \in R_p^m(\lambda, \ell; \mu_1, \mu_2, \gamma)$, where $F_{\mu,p}(f)(z)$ is given by (3.1).

Proof. Let

$$\frac{z\,[I_p^m(\lambda,\ell)(F_{\mu,p}(f)(z))]'}{[I_p^m(\lambda,\ell)(F_{\mu,p}(f)(z))]} = (\gamma - p)\, q(z) - \gamma, \qquad (3.2)$$

where $q(z)$ is analytic in $\mathcal{U}$, $q(0) = 1$, and $q(z) \neq 0$ $(z \in \mathcal{U})$.
Using (3.1), we have

$$z\,[I_p^m(\lambda,\ell)F_{\mu,p}(f)(z))' = \mu\, I_p^m(\lambda,\ell)f(z) - (\mu+p)\, I_p^m(\lambda,\ell)F_{\mu,p}(f)(z). \qquad (3.3)$$

By (3.2) and (3.3), we get

$$\mu \frac{I_p^m(\lambda,\ell)f(z)}{I_p^m(\lambda,\ell)F_{\mu,p}(f)(z)} = (\gamma - p)\, q(z) + [\mu - (\gamma - p)]. \qquad (3.4)$$

Differentiating (3.4) logarithmically, multiplying by z and using (3.2), it follows that

$$\frac{z\,(I_p^m(\lambda,\ell)f(z))'}{I_p^m(\lambda,\ell)f(z)} + \gamma = (\gamma - p)\, q(z) + \frac{(\gamma - p)\, z\, q'(z)}{(\gamma - p)\, q(z) + [\mu - (\gamma - p)]}. \qquad (3.5)$$

The remaining part of the proof is similar to that of Theorem 2.2 and so is omitted.

Theorem 3.2. *If $f \in M_p^m(\lambda, \ell; \mu_1, \mu_2, \gamma)$, and for $\mu > 0$,*

$$-\frac{[z\,[I_p^m(\lambda,\ell)F_{\mu,p}(f)(z)]']'}{[I_p^m(\lambda,\ell)F_{\mu,p}(f)(z)]'} \neq \gamma \quad \forall z \in \mathcal{U},$$

then $F_{\mu,p}(f)(z) \in M_p^m(\lambda, \ell; \mu_1, \mu_2, \gamma)$, where $F_{\mu,p}(f)(z)$ is given by (3.1).

Proof.

$$\begin{aligned} f \in M_p^m(\lambda,\ell;\mu_1,\mu_2,\gamma) &\iff -\tfrac{1}{p} z f' \in R_p^m(\lambda,\ell;\mu_1,\mu_2,\gamma) \\ &\iff F_{\mu,p}\left(-\tfrac{1}{p} z f'(z)\right) \in R_p^m(\lambda,\ell;\mu_1,\mu_2,\gamma) \\ &\iff -\tfrac{1}{p} z\,(F_{\mu,p}(f)(z))' \in R_p^m(\lambda,\ell;\mu_1,\mu_2,\gamma) \\ &\iff F_{\mu,p}(f)(z) \in M_p^m(\lambda,\ell;\mu_1,\mu_2,\gamma). \end{aligned}$$

Letting $\lambda = 1$ in Theorem 3.1, we get the following result:

Corollary 3.3. *If $f \in R_p^m(\ell; \mu_1, \mu_2, \gamma)$, and for $\mu > 0$,*

$$-\frac{z\,[I_p(m,\ell)(F_{\mu,p}(f)(z))]'}{[I_p(m,\ell)(F_{\mu,p}(f)(z))]} \neq \gamma \quad \forall z \in \mathcal{U},$$

then $F_{\mu,p}(f)(z) \in R_p^m(\ell; \mu_1, \mu_2, \gamma)$, where $I_p(m,\ell)f(z)$, $R_p^m(\ell; \mu_1, \mu_2, \gamma)$ and $F_{\mu,p}(f)(z)$ are given by (1.6), (2.14) and (3.1) respectively.

If we set $\ell = 1$ in Theorem 3.1, we can easily get the following result:

Corollary 3.4. *If $f \in R_p^m(\lambda; \mu_1, \mu_2, \gamma)$, and for $\mu > 0$,*

$$-\frac{z\,[D_{\lambda,p}^m(F_{\mu,p}(f)(z))]'}{[D_{\lambda,p}^m(F_{\mu,p}(f)(z))]} \neq \gamma \quad \forall z \in \mathcal{U},$$

then $F_{\mu,p}(f)(z) \in R_p^m(\lambda; \mu_1, \mu_2, \gamma)$, where $D_{\lambda,p}^m f(z)$, $R_p^m(\lambda; \mu_1, \mu_2, \gamma)$ and $F_{\mu,p}(f)(z)$ are given by (1.7), (2.15) and (3.1) respectively.

For $\lambda = 1$ and $\ell = 1$ in Theorem 3.2, we get results similar to Corollary 3.3 and 3.4 respectively.

Acknowledgement: The authors are thankful to worthy referee for his useful suggestions. The first author (S. P. G.) is thankful to CSIR, New Delhi, India for awarding Emeritus Scientistship, under scheme number 21(084)/10/EMR-II. The second author is also thankful to CSIR, India, for providing JRF No. 09/149(0498)/2008-EMR-I.

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