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MORSE-SMALE CHARACTERISTIC IN CIRCLE-VALUED MORSE THEORY

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Abstract. In this paper we consider the $\varphi_{\mathfrak{F}}$ - category associated to the family of circle - valued Morse functions defined on a closed manifold M. This number will be called the Morse-Smale characteristic of manifold M for circle-valued Morse functions and and it will be denoted by $\gamma_{s^1}(M)$. We present some basic notions and results concerning circle-valued Morse functions and we prove some properties for $\gamma_{s^1}(M)$.

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1. Introduction

Consider M^m, N^n two smooth manifolds without boundary. For a mapping $f \in C^\infty(M,N)$ denote by $\mu(f) = |C(f)|$, the cardinal number of critical set $C(f)$ of f. Let $\mathfrak{F} \subseteq C^\infty(M,N)$ be a family of smooth mappings $M \to N$. The $\varphi_{\mathfrak{F}}(M,N)$ category of the pair (M,N) is defined by

$$\varphi_{\mathfrak{F}}(M,N) = \min\{\mu(f) : f \in \mathfrak{F}\}.$$

This notion was introduced by D. Andrica in the paper [1] (see also the monograph [2, pp. 144]). It is clear that $0 \leq \varphi_{\mathfrak{F}}(M,N) \leq +\infty$ and $\varphi_{\mathfrak{F}}(M,N) = 0$ if and only if the family $\mathfrak{F}$ contains immersions, submersions or local diffeomorphisms according to the cases $m < n, m > n$, or $m = n$, respectively. Under some hypotheses $\varphi_{\mathfrak{F}}(M,N)$ is a differential invariant of pair (M,N).

In the monograph [2, pp.146-147] some important particular cases for the family $\mathfrak{F}$ are presented. The main purpose of this note is to define and study the $\varphi_{\mathfrak{F}}$ - category of the pair (M,S^1), where $\mathfrak{F}$ is the family of all smooth circle-valued Morse mappings $f : M \to S^1$.

2. Circle - valued Morse functions

Let M^m be a manifold without boundary, and let $f : M \to S^1$ be a smooth mapping. The circle

$$S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$

is a $1-$dimensional submanifold of $\mathbb{R}^2$ and it is endowed with the induced corresponding smooth structure. Also, the circle S^1 can be identified to the subset of the complex plane consisting in all numbers of the absolute value equal to 1. This identification is useful when we want to prove that the product of two circle-valued functions is also a circle-valued function.

For a point $x \in M$ choose a neighbourhood V of $f(x)$ in S^1 which is diffeomorphic to an open interval of $\mathbb{R}$, and let $U = f^{-1}(V)$. The map $f|_U$ is then identified with a smooth map from U to $\mathbb{R}$. Thus all the local notions of the real-valued Morse theory (see for instance [2], [5] and [8]), in particular the notions of non - degenerate critical point, and the index of a critical point, are carried over. Therefore, a smooth function $f : M \to S^1$ is called *Morse function*, if all its critical points are non - degenerated. For a Morse function $f : M \to S^1$ we denote by $C(f)$ the set of all its critical points, and by $C_k(f)$ the set of all critical points of index k, where $k = 0, ..., m$. It is clear that, if M is compact, then the critical set $C(f)$ is finite; in this case we denote by $\mu(f)$ the cardinality of $C(f)$ and by $\mu_k(f)$ the cardinality of $C_k(f)$, for $k = 0, ..., m$. It is obvious that $\mu(f) = \mu_0(f) + ... + \mu_m(f)$.

The systematic study of circle-valued Morse functions was initiated by S.P.Novikov in 1980. The motivation came from a problem in hydrodinamics, where the application of the variational approach led to a multi-valued Lagrangian. The formulation of the circle-valued Morse theory as a new branch of topology with its own problems and goals was outlined also by S.P.Novikov (see the survey paper [6] for the connection to the theory of closed one-differential forms, [8] and [9]).

The circle S^1 can be also identified with the quotient $\mathbb{R}/\mathbb{Z}$, and circle-valued Morse functions can be considered as multi-valued real functions, so that the value of the function at a point is defined only modulo the subgroup of integers. The natural way to work with the multi-valued functions is to consider a covering space of the domain of definition such that the function becomes single-valued on the covering. Recall the universal covering of the circle

$$exp : \mathbb{R} \to S^1, t \to e^{2\pi i t}.$$

The structure group of this covering is the subgroup $\mathbb{Z} \subset \mathbb{R}$ acting on $\mathbb{R}$ by translations and it is very useful in the circle-valued Morse theory.

3. Morse-Smale characteristic of a manifold for circle-valued Morse functions

The Morse-Smale characteristic of a compact smooth manifold M^m is an important invariant. It is given by $\gamma(M) = min\{\mu(f) : f \in \mathfrak{F}_m(M)\}$, where $\mathfrak{F}_m(M)$ denotes the set of all real-valued Morse functions defined on M. For details, examples and properties we refer to monograph [2, pp.106-129] (see also the papers [3] and [4]). The number $\gamma(M)$ corresponds to the minimal number of cells in the CW-decomposition of M up to a homotopy. This just an important reason why the computation of this invariant is a hard problem in differential topology.

Using the notations from the first section, we consider $N = S^1$ and the family $\mathfrak{F} = \mathfrak{F}_m(M, S^1) \subseteq C^\infty(M, S^1)$, given by the set of all circle-valued Morse functions defined on M. In this case we denote $\varphi_{\mathfrak{F}}(M, S^1)$ by $\gamma_{S^1}(M)$, and we call it the *Morse-Smale characteristic* of manifold M for circle-valued Morse functions $f : M \to S^1$. So, we have

$$\gamma_{S^1}(M) = min\{\mu(f) : f \in \mathfrak{F}_m(M, S^1)\}. \tag{3.1}$$

In an analogous way we define the numbers $\gamma_{S^1}^{(i)}(M)$, for $i = 0, ..., m$, by

$$\gamma_{S^1}^{(i)}(M) = \min\{\mu_i(f) : f \in \mathfrak{F}_m(M, S^1)\}. \tag{3.2}$$

From the relation $\mu(f) = \mu_0(f) + ... + \mu_m(f)$, it follows that for any $f \in \mathfrak{F}_m(M, S^1)$ we have $\mu(f) \geq \gamma_{S^1}^{(0)}(M) + ... + \gamma_{S^1}^{(m)}(M)$. Therefore the following inequality holds:

$$\gamma_{S^1}(M) \geq \sum_{i=0}^{m} \gamma_{S^1}^{(i)}(M). \tag{3.3}$$

Now we will show that $\gamma_{S^1}(M)$ and $\gamma_{S^1}^{(i)}(M)$ are differential invariants of manifold M for $i = 0, ..., m$. For this purpose first of all let us consider a smooth manifold N, $\varphi : M \to N$ a diffeomorphism and $f : M \to S^1$, $g : M \to S^1$ two smooth mappings such that $g = f \circ \varphi$. Then we have the relation $C(f) = \varphi(C(g))$ (see [2, pp.107] for the case of real mappings). With a similar proof as in [2, Lemma 4.1.3, pp.107] we obtain the following property : If $f \in \mathfrak{F}_m(M, S^1)$, then $g \in \mathfrak{F}_m(M, S^1)$ and the corresponding critical points via the diffeomorphism φ are of the same Morse index.

Theorem 3.1. *If the manifolds M, N are diffeomorphic, then* $\gamma_{S^1}(M) = \gamma_{S^1}(N)$ *and* $\gamma_{S^1}^{(i)}(M) = \gamma_{S^1}^{(i)}(N)$*, for* $i = 0, ..., m$*. That is, these numbers are differential invariants of the manifold.*

Proof. Let $\varphi : M \to N$ be a diffeomophism between M and N, and let $\Gamma : \mathfrak{F}_m(M, S^1) \to \mathfrak{F}_m(N, S^1)$ be the function defined by $\Gamma(f) = f \circ \varphi$. Taking into the account the comments given before the theorem, it follows that Γ is well-defined. Consider the function $\Lambda : \mathfrak{F}_m(N,S^1) \to \mathfrak{F}_m(M,S^1)$, where $\Lambda(g) = g \circ \varphi^{-1}$ and observe that $\Gamma \circ \Lambda = 1_{\mathfrak{F}_m(N,S^1)}$ and $\Lambda \circ \Gamma = 1_{\mathfrak{F}_m(M,S^1)}$, thus Γ is a natural bijection. We can write

$$\gamma_{S^1}(M) = min\{\mu(f) : f \in \mathfrak{F}_m(M,S^1)\} = min\{\mu(f\circ\varphi) : f \in \mathfrak{F}_m(M,S^1)\}$$
$$= min\{\mu(\Gamma(f)) : f \in \mathfrak{F}_m(M,S^1)\} = min\{\mu(g) : g \in \mathfrak{F}_m(N,S^1)\} = \gamma_{S^1}(N).$$

In an analogous way we can prove the equalities $\gamma^{(i)}_{S^1}(M) = \gamma^{(i)}_{S^1}(N)$, for $i = 0,...,m$. □

Theorem 3.2. *The following relations hold:*
(1) (Symmetry)For any $i = 0,...,m$, we have

$$\gamma^{(i)}_{S^1}(M) = \gamma^{(m-i)}_{S^1}(M); \tag{3.4}$$

(2) (Submultiplicity) For any two manifolds M and N, we have:

$$\gamma_{S^1}(M \times N) \leq \gamma_{S^1}(N) \times \gamma_{S^1}(N); \tag{3.5}$$

(3) For any $i = 0,...,m+n$, we have:

$$\gamma^{(i)}_{S^1}(M \times N) \leq \sum_{j+k=i} \gamma^{(j)}_{S^1}(M)\cdot\gamma^{(k)}_{S^1}(N). \tag{3.6}$$

Proof. (1) For a circle-valued Morse function $f \in \mathfrak{F}_m(M,S^1)$ the mapping $h = -f$ satisfies the relations $h \in \mathfrak{F}_m(M,S^1)$, and for any $i = 0,...,m$, $\mu_i(f) = \mu_{m-i}(h)$. We get that for any circle-valued Morse function f on manifold M, we have $\mu_i(f) \geq \gamma^{(m-i)}_{S^1}(M)$ and consequently $\gamma^{(i)}_{S^1}(M) \geq \gamma^{(m-i)}_{S^1}(M)$. Replacing i by $m-i$, from the above inequalities it follows $\gamma^{(m-i)}_{S^1}(M) \geq \gamma^{(i)}_{S^1}(M)$, and we are done.

(2) Consider the circle-valued Morse functions $f \in \mathfrak{F}_m(M,S^1)$ and $g \in \mathfrak{F}_m(N,S^1)$ and let $h : M \times N \to S^1$ be the mapping defined by $h(x,y) = f(x)g(y)$. It is easy to see that $C(h) = C(f) \times C(g)$, thus we have $\mu(h) = \mu(f)\mu(g)$. On the other hand, after an elementary computation, it follows that the Hessian matrix $H(h)(p,q)$ of the locally representation of h around a critical point (p,q) is of the form

$$\left(\begin{array}{c|c} \mathrm{g(q)H(f)(p)} & 0 \\ \hline 0 & \mathrm{f(p)H(g)(q)} \end{array}\right)$$

where $H(f)(p)$ and $H(g)(q)$ are the Hessian matrices of the locally representations of f and g, respectively. We can assume that $f(p) \neq 0$ and $g(q) \neq 0$, hence $H(h)(p,q)$

is non-singular since both $H(f)(p)$ and $H(g)(q)$ are non-singular. It follows that h is a circle-valued Morse function on $M \times N$. From the relation $\mu(h) = \mu(f)\mu(g)$ one obtains $\gamma_{S^1}(M \times N) \leq \mu(f)\mu(g)$ for any circle-valued Morse functions $f \in \mathfrak{F}_m(M, S^1)$, and $g \in \mathfrak{F}_m(N, S^1)$. That is $\gamma_{S^1}(M \times N) \leq \gamma_{S^1}(M) \times \gamma_{S^1}(N)$.

(3) With the above notations, we can assume that the locally representations of f and g satisfy $f > 0$ and $g > 0$. Therefore, if p is critical point of index j of f and q is a critical point of index k of g, then (p, q) is a critical point of index $j + k$ of h. That is, we have the following relation

$$\mu_i(h) = \sum_{j+k=i} \mu_j(f)\mu_k(g),$$

for an circle-valued Morse functions $f \in \mathfrak{F}_m(M, S^1)$, and $g \in \mathfrak{F}_m(N, S^1)$. According to the definition of the numbers $\gamma_{S^1}^{(i)}$, the desired relations follow. □

References

[1] Andrica, D., *Functions with minimal critical set: new results and open problems*, Mathematical Analysis and Applications, Th.M.Rassias ed., Hadronic Press, 1999, 1-10.

[2] Andrica D., *Critical Points Theory and Some Applications*, Cluj University Press, Cluj - Napoca, 2005.

[3] Andrica D., *Some Remarks Concerning the Morse-Smale Characteristic of a Compact Manifold*, Mathematica, L'Analyse Numerique et la Theorie de l'Approximation, Tome 21, no.1, 1992, 9-13.

[4] Andrica D., *The Morse-Smale Characteristic of a Simply-Connected Compact Manifold*, Mathematica, L'Analyse Numerique et la Theorie de l'Approximation, Tome 22, no.2, 1993, 121-124.

[5] Cicortas G., *Elements of Morse Theory and Applications* (Romanian), Cluj University Press, Cluj - Napoca, 2008.

[6] Farber, M., *Topology of Closed One-Forms*, Mathematical Surveys and Monographs, Volume 108, American Mathematical Society, 2004.

[7] Greenberg, M.J.,Harper, J.R., *Algebraic Topology. A First Course*, Addison-Wesley,1981.

[8] Matsumoto Y., *An Introduction to Morse Theory*, Translations of Mathematical Monographs, Vol. 208, Iwanami Series in Modern Mathematics, American Mathematical Society, Providence, Rhode Island.

[9] Pajitnov A., *Circle - Valued Morse Theory*, Walter de Gruyter, Berlin - New York, 2006.

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