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INTEGRAL OPERATORS ON A CERTAIN CLASS OF UNIVALENT FUNCTIONS

Virgil Pescar

Abstract. In this work is considered the class $\mathcal{T}_2$ of univalent functions defined by the condition $\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| < 1$ for $|z| < 1$, where $f(z) = z + a_3 z^3 + ...$ is analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. The integral operators G_γ, J_γ, $J_{\gamma_1,\gamma_2,...,\gamma_n}$, $D_{\alpha,\beta}$, $L_{\alpha,\beta}$, $K_{\gamma_1,\gamma_2,...,\gamma_n}$ and $H_{\gamma_1,\gamma_2,...,\gamma_n,\beta,\delta}$, for the functions $f \in \mathcal{T}_2$ are considered. In the present paper we obtain univalence conditions of these integral operators.

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1. Introduction

Let $\mathcal{A}$ be the class of the functions f which are analitic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ and $f(0) = f'(0) - 1 = 0$.

We denote by $\mathcal{S}$ the class of the functions $f \in \mathcal{A}$ which are univalent in $\mathcal{U}$.

We consider the integral operators

$$G_\gamma(z) = \int_0^z \left(\frac{f(u)}{u}\right)^{\frac{1}{\gamma}} du, \tag{1.1}$$

$$J_\gamma(z) = \left[\frac{1}{\gamma}\int_0^z u^{-1}(f(u))^{\frac{1}{\gamma}} du\right]^\gamma, \tag{1.2}$$

$$J_{\gamma_1,\gamma_2,...,\gamma_n}(z) = \left[\sum_{j=1}^n \frac{1}{\gamma_j}\int_0^z u^{-1}\prod_{j=1}^n (f_j(u))^{\frac{1}{\gamma_j}} du\right]^{\frac{1}{\sum_{j=1}^n \frac{1}{\gamma_j}}}, \tag{1.3}$$

$$D_{\alpha,\beta}(z) = \left[\beta \int_0^z u^{\beta-1}\left(\frac{f(u)}{u}\right)^{\frac{1}{\alpha}} du\right]^{\frac{1}{\beta}}, \tag{1.4}$$

$$L_{\alpha,\beta}(z) = \left[\beta \int_0^z u^{\beta-1}\prod_{j=1}^{n}\left(\frac{f_j(u)}{u}\right)^{\frac{1}{\alpha}} du\right]^{\frac{1}{\beta}}, \tag{1.5}$$

$$K_{\gamma_1,\gamma_2,\dots,\gamma_n}(z) = \int_0^z \prod_{j=1}^{n}\left(\frac{f_j(u)}{u}\right)^{\frac{1}{\gamma_j}} du, \tag{1.6}$$

for $f \in \mathcal{A}$, α, β, γ complex numbers, $\alpha \neq 0$, $\beta \neq 0$, $\gamma \neq 0$ and $f_j \in \mathcal{A}$, γ_j complex numbers, $\gamma_j \neq 0$, $j = \overline{1,n}$.

In [1], [2], [4], [7], [8], [9], [10], [11] we have certain the univalence conditions of these integral operators.

We define a general integral operator

$$H_{\gamma_1,\gamma_2,\dots,\gamma_n,\beta,\delta}(z) = \left[\beta\delta \int_0^z u^{\beta\delta-1}\prod_{j=1}^{n}\left(\frac{f_j(u)}{u}\right)^{\frac{1}{\gamma_j}} du\right]^{\frac{1}{\beta\delta}}, \tag{1.7}$$

for $f_j \in \mathcal{A}$, β, δ, γ_j complex numbers, $\beta\delta \neq 0$, $\gamma_j \neq 0$, $j = \overline{1,n}$, $n \in \mathbb{N} - \{0\}$.

For β, δ, γ_j, $n \in \mathbb{N} - \{0\}$, $j = \overline{1,n}$, in the particular cases, from (1.7) we obtain the integral operators G_γ, J_γ, $J_{\gamma_1,\gamma_2,\dots,\gamma_n}$, $D_{\alpha,\beta}$, $L_{\alpha,\beta}$, $K_{\gamma_1,\gamma_2,\dots,\gamma_n}$.

2. Preliminary results

We need the following theorems.

Theorem 2.1. **[6]**.*Let α be a complex number,* $\text{Re}\alpha > 0$ *and $f \in \mathcal{A}$. If*

$$\frac{1-|z|^{2\text{Re}\alpha}}{\text{Re}\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1, \tag{2.1}$$

for all $z \in \mathcal{U}$, then for any complex number β, $\text{Re}\beta \geq \text{Re}\alpha$*, the function*

$$F_\beta(z) = \left[\beta \int_0^z u^{\beta-1} f'(u)\, du\right]^{\frac{1}{\beta}} \tag{2.2}$$

is in the class $\mathcal{S}$.

Theorem 2.2. (Schwarz [3]). *Let f be the function regular in the disk $\mathcal{U}_R = \{z \in \mathbb{C} : |z| < R\}$ with $|f(z)| < M$, M fixed. If $f(z)$ has in $z = 0$ one zero with multiply $\geq m$, then*

$$|f(z)| \leq \frac{M}{R^m} |z|^m , \ (z \in \mathcal{U}_R) , \tag{2.3}$$

the equality (in the inequality (2.3) for $z \neq 0$) can hold only if

$$f(z) = e^{i\theta} \frac{M}{R^m} z^m,$$

where θ is constant.

Theorem 2.3. [5]. *Assume that the function $f \in \mathcal{A}$ satisfies the condition*

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| < 1, \ \ (z \in \mathcal{U}), \tag{2.4}$$

then the function f is univalent in $\mathcal{U}$.

3. Main results

Theorem 3.1. *Let γ_j, α complex numbers, $\gamma_j \neq 0$, $j = \overline{1,n}$, $\mathrm{Re}\alpha > 0$, M_j positive real numbers and $f_j \in \mathcal{T}_2$, $f_j(z) = z + \sum\limits_{k=3}^{\infty} a_{kj} z^k$, $j = \overline{1,n}$, $n \in \mathbb{N} - \{0\}$.*

If

$$|f_j(z)| \leq M_j, \quad (j = \overline{1,n}; \ z \in \mathcal{U}) \tag{3.1}$$

and

$$\sum_{j=1}^{n} \frac{2M_j + 1}{|\gamma_j|} \leq \mathrm{Re}\alpha, \tag{3.2}$$

then for any complex numbers β and δ, $\mathrm{Re}\beta\delta \geq \mathrm{Re}\alpha$, the function

$$H_{\gamma_1,\gamma_2,\ldots,\gamma_n,\beta,\delta}(z) = \left\{ \beta\delta \int_0^z u^{\beta\delta - 1} \left(\frac{f_1(u)}{u} \right)^{\frac{1}{\gamma_1}} \ldots \left(\frac{f_n(u)}{u} \right)^{\frac{1}{\gamma_n}} du \right\}^{\frac{1}{\beta\delta}} \tag{3.3}$$

is in the class $\mathcal{S}$.

Proof. We consider the function

$$h(z)=\int\limits_0^z \left(\frac{f_1(u)}{u}\right)^{\frac{1}{\gamma_1}}\ldots\left(\frac{f_n(u)}{u}\right)^{\frac{1}{\gamma_n}}du. \tag{3.4}$$

The function h is regular in $\mathcal{U}$.
We have

$$\left|\frac{zh''(z)}{h'(z)}\right|=\sum_{j=1}^{n}\frac{1}{|\gamma_j|}\left|\frac{zf_j'(z)}{f_j(z)}-1\right|,\ (z\in\mathcal{U}). \tag{3.5}$$

We obtain

$$\left|\frac{zf_j'(z)}{f_j(z)}-1\right|\leq\left|\frac{z^2f_j'(z)}{f_j^2(z)}\right|\left|\frac{f_j(z)}{z}\right|+1\leq$$

$$\leq\left|\frac{z^2f_j'(z)}{f_j^2(z)}-1\right|\frac{|f_j(z)|}{|z|}+\frac{|f_j(z)|}{|z|}+1,\ (j=\overline{1,n};\ z\in\mathcal{U}). \tag{3.6}$$

Since $f_j\in\mathcal{T}_2$ and by Theorem 2.2, from (3.6) we get

$$\left|\frac{zf_j'(z)}{f_j(z)}-1\right|\leq 2M_j+1\left(j=\overline{1,n};\ z\in\mathcal{U}\right). \tag{3.7}$$

From (3.5) and (3.7) we obtain

$$\frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}\left|\frac{zh''(z)}{h'(z)}\right|\leq\frac{1-|z|^{2\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}\sum_{j=1}^{n}\frac{2M_j+1}{|\gamma_j|},\ (z\in\mathcal{U}) \tag{3.8}$$

and hence, by (3.2) we have

$$\frac{1-|z|^{\mathrm{Re}\alpha}}{\mathrm{Re}\alpha}\left|\frac{zh''(z)}{h'(z)}\right|\leq 1, \tag{3.9}$$

for all $z\in\mathcal{U}$.

So, by Theorem 2.1, the integral operator $H_{\gamma_1,\gamma_2,\ldots,\gamma_n,\beta,\delta}$ is in the class $\mathcal{S}$.

Corollary 3.2. *Let* γ_j*,* α *complex numbers,* $\mathrm{Re}\gamma_j \neq 0$*,* $j = \overline{1,n}$*,* $\sum\limits_{j=1}^{n} \mathrm{Re}\frac{1}{\gamma_j} \geq \mathrm{Re}\alpha > 0$*,* M_j *positive real numbers and* $f_j \in \mathcal{T}_2$*,* $f_j(z) = z + a_{3j}z^3 + \ldots$*,* $j = \overline{1,n}$*,* $n \in \mathbb{N} - \{0\}$*.*

If

$$|f_j(z)| \leq M_j, \ \ (j = \overline{1,n};\ z \in \mathcal{U}) \tag{3.10}$$

and

$$\sum_{j=1}^{n} \frac{2M_j + 1}{|\gamma_j|} \leq \mathrm{Re}\alpha, \tag{3.11}$$

then the integral operator $J_{\gamma_1,\gamma_2,\ldots,\gamma_n}$ *given by (1.3) is in the class* $\mathcal{S}$*.*

Proof. For $\beta\delta = \sum\limits_{j=1}^{n} \frac{1}{\gamma_j}$ from Theorem 3.1 we obtain Corollary 3.2.

Remark 3.3. From Corollary 3.2, for $n = 1$, $\gamma_1 = \gamma$, $f_1 = f$, we obtain the integral operator J_γ defined by (1.2) is in the class $\mathcal{S}$.

Corollary 3.4. *Let* γ_j*,* α *complex numbers,* $\gamma_j \neq 0$*,* $j = \overline{1,n}$*,* $0 < \mathrm{Re}\alpha \leq 1$*,* M_j *positive real numbers and* $f_j \in \mathcal{T}_2$*,* $f_j(z) = z + a_{3j}z^3 + \ldots$*,* $j = \overline{1,n}$*,* $n \in \mathbb{N} - \{0\}$*.*

If

$$|f_j(z)| \leq M_j,\ j = \overline{1,n},\ (z \in \mathcal{U}) \tag{3.12}$$

and

$$\sum_{j=1}^{n} \frac{2M_j + 1}{|\gamma_j|} \leq \mathrm{Re}\alpha, \tag{3.13}$$

then the function

$$K_{\gamma_1,\gamma_2,\ldots,\gamma_n}(z) = \int_0^z \left(\frac{f_1(u)}{u}\right)^{\frac{1}{\gamma_1}} \ldots \left(\frac{f_n(u)}{u}\right)^{\frac{1}{\gamma_n}} du \tag{3.14}$$

is in the class $\mathcal{S}$*.*

Proof. For $\beta\delta = 1$, from Theorem 3.1, we obtain Corollary 3.4.

Remark 3.5. If we take $n = 1$, $\gamma_1 = \gamma$, $f_1 = f$, from Corollary 3.4 we have the integral operator G_γ given by (1.1) is in the class $\mathcal{S}$.

Corollary 3.6. *Let α, γ complex numbers, $\alpha \neq 0$, $\mathrm{Re}\gamma > 0$, M_j positive real numbers and $f_j \in \mathcal{T}_2$, $f_j(z) = z + a_{3j}z^3 + ...$, $j = \overline{1,n}$, $n \in \mathbb{N} - \{0\}$.*

If

$$|f_j(z)| \leq M_j,\ (j = \overline{1,n},\ z \in \mathcal{U}), \tag{3.15}$$

and

$$\sum_{j=1}^{n} \frac{2M_j + 1}{|\alpha|} \leq \mathrm{Re}\gamma, \tag{3.16}$$

then for any complex number β, $\mathrm{Re}\beta \geq \mathrm{Re}\gamma$, the function

$$L_{\alpha,\beta}(z) = \left\{ \beta \int_0^z u^{\beta-1} \left(\frac{f_1(u)}{u} \right)^{\frac{1}{\alpha}} ... \left(\frac{f_n(u)}{u} \right)^{\frac{1}{\alpha}} du \right\}^{\frac{1}{\beta}} \tag{3.17}$$

is in the class $\mathcal{S}$.

Proof. For $\delta = 1$, from Theorem 3.1, we have Corollary 3.6.

Remark 3.7. If take $n = 1$, $f_1 = f$ in Corollary 3.6, we obtain that the integral operator $D_{\alpha,\beta}$ defined by (1.4) is in the class $\mathcal{S}$.

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Virgil Pescar
Department of Mathematics
"Transilvania" University of Braşov
Faculty of Mathematics and Computer Science
500091 Braşov, Romania
email: *virgilpescar@unitbv.ro*