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THE (G'/G)-EXPANSION METHOD FOR SOLVING THE COMBINED AND THE DOUBLE COMBINED SINH-COSH-GORDON EQUATIONS

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Abstract. In this paper, the (G'/G)-expansion method is applied to seek traveling wave solutions of the combined and the double combined sinh-cosh-Gordon equations. This traveling wave solutions are expressed by the hyperbolic functions and the trigonometric functions. It is shown that the proposed method is direct, effective and more general.

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1. Introduction

The sinh-Gordon equation,

$$u_{tt} - u_{xx} + \sinh u = 0, \tag{1}$$

gained its importance because of the kink and antikink solutions with the collisional behaviors of solitons that arise from this equation. The equation appears in integrable quantum field theory, kink dynamics, and fluid dynamics [1-8].

Many powerful methods, such as Bäcklund transformation, inverse scattering method, Hirota bilinear forms, pseudo spectral method, the tanh method, the tanh-sech method, the sine-cosine method [9-12], and many others were successfully used to nonlinear equations. Recently, Wang et al. [15] proposed the (G'/G)-expansion method and showed that it is powerful for finding analytic solutions of PDEs. Next, Bekir [16] applied the method to some nonlinear evolution equations gaining traveling wave solutions. More recently, Zhang et al. [17] proposed a generalized $(\frac{G'}{G})$-expansion method to improve and extend Wang et al.'s work [15] for solving variable coefficient equations and high dimensional equations. Also Zhang [17] solved the equations with the balance numbers of which are not positive integers, by this method. In this paper we will apply the (G'/G)-expansion method to the combined sinh-cosh-Gordon and double combined sinh-cosh-Gordon equations.

2.Description of the (G'/G)-expansion method

We suppose that the given nonlinear partial differential equation for $u(x,t)$ to be in the form

$$P(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, ...) = 0, \quad (2)$$

where P is a polynomial in its arguments. The essence of the (G'/G)-expansion method can be presented in the following steps:

step 1. Seek traveling wave solutions of Eq. (2) by taking $u(x,t) = U(\xi)$, $\xi = x-ct$, and transform Eq. (2) to the ordinary differential equation

$$Q(U, U', U'', ...) = 0, \quad (3)$$

where prime denotes the derivative with respect to ξ.

step 2. If possible, integrate Eq. (3) term by term one or more times. This yields constant(s) of integration. For simplicity, the integration constant(s) can be set to zero.

step 3. Introduce the solution $U(\xi)$ of Eq. (3) in the finite series form

$$U(\xi) = \sum_{i=0}^{N} a_i (\frac{G'(\xi)}{G(\xi)})^i, \quad (4)$$

where a_i are real constants with $a_N \neq 0$ to be determined, N is a positive integer to be determined. The function $G(\xi)$ is the solution of the auxiliary linear ordinary differential equation

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \quad (5)$$

where λ and μ are real constants to be determined.

step 4. Determine N. This, usually, can be accomplished by balancing the linear term(s) of highest order with the highest order nonlinear term(s) in Eq. (3).

step 5. Substituting (4) together with (5) into Eq. (3) yields an algebraic equation involving powers of (G'/G). Equating the coefficients of each power of (G'/G) to zero gives a system of algebraic equations for a_i, λ, μ and c. Then, we solve the system with the aid of a computer algebra system, such as Maple, to determine these constants. On the other hand, depending on the sign of the discrimmant $\Delta = \lambda^2 - 4\mu$, the solutions of Eq. (5) are well known to us. So, as a final step, we can obtain exact solutions of the given Eq. (2).

3.The combined sinh-cosh-Gordon equation

We first solve the combined sinh-cosh-Gordon equation

$$u_{tt} - ku_{xx} + \alpha \sinh u + \beta \cosh u = 0, \quad (6)$$

where α and β are nonzero real constants. Making the transformation $u(x,t) = u(\xi)$, $\xi = x - ct$ and integrating once with respect to ξ, Eq. (6) we get

$$(c^2 - k)u'' + \alpha \sinh u + \beta \cosh u = 0. \tag{7}$$

By applying the Painlevé transformation,

$$v = e^u, \tag{8}$$

or equivalently

$$u = \ln v. \tag{9}$$

we have

$$\sinh u = \frac{v - v^{-1}}{2}, \quad \cosh u = \frac{v + v^{-1}}{2}. \tag{10}$$

Then

$$u = \operatorname{arccosh}[\frac{v + v^{-1}}{2}]. \tag{11}$$

Consequently, we can write the combined sinh-cosh-Gordon equation (7) to the ODE

$$(\alpha + \beta)v^3 - (\alpha - \beta)v + 2(c^2 - k)vv'' - 2(c^2 - k)(v')^2 = 0. \tag{12}$$

Balancing the v^3 with vv'' gives

$$M = 2. \tag{13}$$

The (G'/G)-expansion method allows us to use the finite expansion

$$v(\xi) = a_0 + a_1(\frac{G'}{G}) + a_2(\frac{G'}{G})^2, \qquad a_2 \neq 0. \tag{14}$$

Substituting (14) into (12), setting coefficients of $(\frac{G'}{G})^i (i = 0, 1, ..., 6)$ to zero, we obtain the following under-determined system of algebraic equations for a_0, a_1, a_2, c, λ and μ:

$$\begin{aligned}
(\frac{G'}{G})^0 \ &: \ \alpha {a_0}^3 + \beta {a_0}^3 + 4c^2 a_0 a_2 \mu^2 - 4k a_0 a_2 \mu^2 - \alpha a_0 + \beta a_0 + 2c^2 a_0 a_1 \lambda \mu \\
&\quad -2k a_0 a_1 \lambda \mu - 2c^2 {a_1}^2 \mu^2 + 2k {a_1}^2 \mu^2 = 0, \\
(\frac{G'}{G})^1 \ &: \ -\alpha a_1 + \beta a_1 + 3\alpha a_1 {a_0}^2 + 3\beta a_1 {a_0}^2 + 2c^2 a_0 a_1 \lambda^2 + 4c^2 a_0 a_1 \mu \\
&\quad +12c^2 a_0 a_2 \lambda \mu - 2c^2 {a_1}^2 \lambda \mu - 4c^2 a_1 a_2 \mu^2 - 2k a_0 a_1 \lambda^2 - 4k a_0 a_1 \mu \\
&\quad -12k a_0 a_2 \lambda \mu + 2k {a_1}^2 \lambda \mu + 4k a_1 a_2 \mu^2 = 0, \\
(\frac{G'}{G})^2 \ &: \ -\alpha a_2 + \beta a_2 + 3\alpha a_2 {a_0}^2 + 3\alpha {a_1}^2 a_0 + 3\beta a_2 {a_0}^2 + 3\beta {a_1}^2 a_0 + 6c^2 a_0 a_1 \lambda
\end{aligned}$$

$$
\begin{aligned}
& +8c^2a_0a_2\lambda^2+16c^2a_0a_2\mu-2c^2a_1a_2\lambda\mu-4c^2{a_2}^2\mu^2-6ka_0a_1\lambda \\
& -8ka_0a_2\lambda^2-16ka_0a_2\mu+2ka_1a_2\lambda\mu+4k{a_2}^2\mu^2=0, \\
(\frac{G'}{G})^3 \quad : \quad & \alpha{a_1}^3+\beta{a_1}^3+6\alpha a_1a_2a_0+6\beta a_1a_2a_0+4c^2a_0a_1+20c^2a_0a_2\lambda \\
& +2c^2{a_1}^2\lambda+2c^2a_1a_2\lambda^2+4c^2a_1a_2\mu-4c^2{a_2}^2\lambda\mu-4ka_0a_1 \\
& -20ka_0a_2\lambda-2k{a_1}^2\lambda-2ka_1a_2\lambda^2-4ka_1a_2\mu+4k{a_2}^2\lambda\mu=0, \\
(\frac{G'}{G})^4 \quad : \quad & 2c^2{a_1}^2-2k{a_1}^2+3\alpha{a_2}^2a_0+3\alpha{a_1}^2a_2+3\beta{a_2}^2a_0+3\beta{a_1}^2a_2 \\
& +12c^2a_0a_2+10c^2a_1a_2\lambda-12ka_0a_2-10ka_1a_2\lambda=0, \\
(\frac{G'}{G})^5 \quad : \quad & 3\alpha a_1{a_2}^2+3\beta a_1{a_2}^2+8c^2a_1a_2+4c^2{a_2}^2\lambda-8ka_1a_2-4ka_2^2\lambda=0, \\
(\frac{G'}{G})^6 \quad : \quad & \alpha{a_2}^3+\beta{a_2}^3+4c^2{a_2}^2-4k{a_2}^2=0.
\end{aligned}
$$

Solving this system by Maple, gives

- $$a_0=\pm\sqrt{\frac{\alpha-\beta}{\alpha+\beta}}\frac{\lambda^2}{\lambda^2-4\mu},\quad a_1=\pm\sqrt{\frac{\alpha-\beta}{\alpha+\beta}}\frac{4\lambda}{\lambda^2-4\mu},\quad a_2=\pm\sqrt{\frac{\alpha-\beta}{\alpha+\beta}}\frac{4}{\lambda^2-4\mu},$$
$$c=\pm\sqrt{k\mp\frac{\sqrt{\alpha^2-\beta^2}}{\lambda^2-4\mu}},\quad \alpha>\beta,\quad c^2>k, \tag{15}$$

where λ and μ are arbitrary constants. Substituting Eq. (15) into Eq. (14) yields

$$v(\xi) = \pm\sqrt{\frac{\alpha-\beta}{\alpha+\beta}}\frac{\lambda^2}{\lambda^2-4\mu}\pm\sqrt{\frac{\alpha-\beta}{\alpha+\beta}}\frac{4\lambda}{\lambda^2-4\mu}(\frac{G'}{G})\pm\sqrt{\frac{\alpha-\beta}{\alpha+\beta}}\frac{4}{\lambda^2-4\mu}(\frac{G'}{G})^2, \tag{16}$$

where $\xi=x-(\pm\sqrt{k\mp\frac{\sqrt{\alpha^2-\beta^2}}{\lambda^2-4\mu}})t$.
Substituting general solutions of Eq. (5) into Eq. (16), we have two types of traveling wave solutions of the combined sinh-cosh-Gordon equation as follows:
When $\lambda^2-4\mu>0$,

$$v_1(\xi) = \pm\sqrt{\frac{\alpha-\beta}{\alpha+\beta}}(\frac{c_1\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi+c_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{c_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi+c_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi})^2, \tag{17}$$

from (11), we therefore obtain the solutions

$$u_1(\xi) = \operatorname{arccosh}(\frac{1}{2}\{\pm\sqrt{\frac{\alpha-\beta}{\alpha+\beta}}(\frac{c_1\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi+c_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{c_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi+c_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi})^2$$

$$\pm\sqrt{\frac{\alpha+\beta}{\alpha-\beta}}(\frac{c_1\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi+c_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{c_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi+c_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi})^{-2}\}), \quad (18)$$

where $\xi = x - (\pm\sqrt{k \mp \frac{\sqrt{\alpha^2-\beta^2}}{\lambda^2-4\mu}})t$.
When $\lambda^2 - 4\mu < 0$,

$$v_2(\xi) = \mp\sqrt{\frac{\alpha-\beta}{\alpha+\beta}}(\frac{-c_1\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi+c_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{c_1\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi+c_2\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi})^2, \quad (19)$$

from (11), we therefore obtain the solutions

$$u_2(\xi) = \operatorname{arccosh}(\frac{1}{2}\{\mp\sqrt{\frac{\alpha-\beta}{\alpha+\beta}}(\frac{-c_1\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi+c_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{c_1\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi+c_2\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi})^2$$
$$\mp\sqrt{\frac{\alpha+\beta}{\alpha-\beta}}(\frac{-c_1\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi+c_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{c_1\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi+c_2\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi})^{-2}\}), \quad (20)$$

where $\xi = x - (\pm\sqrt{k \mp \frac{\sqrt{\alpha^2-\beta^2}}{\lambda^2-4\mu}})t$.

In solutions (18) and (20), c_1 and c_2 are left as free parameters.

In particular, if $c_1 \neq 0$ and $c_2 = 0$, then u_1 becomes

$$u_1(\xi) = \operatorname{arccosh}(\pm\frac{1}{2}\sqrt{\frac{\alpha-\beta}{\alpha+\beta}}\tanh^2[\frac{\sqrt[4]{\alpha^2-\beta^2}}{2\sqrt{c^2-k}}(x-ct)]$$
$$\pm\frac{1}{2}\sqrt{\frac{\alpha+\beta}{\alpha-\beta}}\coth^2[\frac{\sqrt[4]{\alpha^2-\beta^2}}{2\sqrt{c^2-k}}(x-ct)]), \quad (21)$$

if $c_2 \neq 0$ and $c_1 = 0$,

$$u_1(\xi) = \operatorname{arccosh}(\pm\frac{1}{2}\sqrt{\frac{\alpha-\beta}{\alpha+\beta}}\coth^2[\frac{\sqrt[4]{\alpha^2-\beta^2}}{2\sqrt{c^2-k}}(x-ct)]$$
$$\pm\frac{1}{2}\sqrt{\frac{\alpha+\beta}{\alpha-\beta}}\tanh^2[\frac{\sqrt[4]{\alpha^2-\beta^2}}{2\sqrt{c^2-k}}(x-ct)]), \quad (22)$$

where $\alpha > \beta$ and $c^2 > k$, which are the solitary solutions of the combined sinh-cosh-Gordon equation.

If $c_1 \neq 0$, $c_2 = 0$, then u_2 becomes

$$u_2(\xi) = \operatorname{arccosh}(\pm\frac{1}{2}\sqrt{\frac{\alpha-\beta}{\alpha+\beta}}\tan^2[\frac{\sqrt[4]{\alpha^2-\beta^2}}{2\sqrt{k-c^2}}(x-ct)]$$
$$\mp\frac{1}{2}\sqrt{\frac{\alpha+\beta}{\alpha-\beta}}\cot^2[\frac{\sqrt[4]{\alpha^2-\beta^2}}{2\sqrt{k-c^2}}(x-ct)]), \quad (23)$$

if $c_2 \neq 0$, $c_1 = 0$,

$$u_2(\xi) = \operatorname{arccosh}(\mp\frac{1}{2}\sqrt{\frac{\alpha-\beta}{\alpha+\beta}}\cot^2[\frac{\sqrt[4]{\alpha^2-\beta^2}}{2\sqrt{k-c^2}}(x-ct)]$$
$$\mp\frac{1}{2}\sqrt{\frac{\alpha+\beta}{\alpha-\beta}}\tan^2[\frac{\sqrt[4]{\alpha^2-\beta^2}}{2\sqrt{k-c^2}}(x-ct)]), \quad (24)$$

where $\alpha > \beta$ and $k > c^2$, which are the periodic solutions of the combined sinh-cosh-Gordon equation.

The solutions (21)-(24) are same Eq. (28)- Eq. (31) in [18] respectively. Therefore the solutions in [18] are only a special case of the our solutions.

4.The double combined sinh-cosh-Gordon equation

In this section we consider the double combined sinh-cosh-Gordon equation

$$u_{tt} - ku_{xx} + \alpha\sinh u + \alpha\cosh u + \beta\sinh(2u) + \beta\cosh(2u) = 0, \quad (25)$$

where α and β are nonzero real constants. Making the transformation $u(x,t) = u(\xi)$, $\xi = x - ct$ and integrating once with respect to ξ, we get

$$(c^2-k)u'' + \alpha\sinh u + \alpha\cosh u + \beta\sinh(2u) + \beta\cosh(2u) = 0. \quad (26)$$

Using the transformation

$$v = e^u, \quad (27)$$

or equivalently

$$u = \ln v. \quad (28)$$

We have

$$\sinh(u) = \frac{v-v^{-1}}{2}, \quad \sinh(2u) = \frac{v^2-v^{-2}}{2},$$

$$\cosh(u) = \frac{v + v^{-1}}{2}, \quad \cosh(2u) = \frac{v^2 + v^{-2}}{2}. \tag{29}$$

Then

$$u = \operatorname{arccosh}[\frac{v + v^{-1}}{2}]. \tag{30}$$

Consequently, we can write the double combined sinh-cosh-Gordon (26) in ODE form

$$2\beta v^4 + 2\alpha v^3 + 2(c^2 - k)vv'' - 2(c^2 - k)(v')^2 = 0. \tag{31}$$

Balancing the v^4 with vv'' gives

$$M = 1. \tag{32}$$

Proceeding as before, we use the finite expansion

$$v(\xi) = a_0 + a_1(\frac{G'}{G}), \qquad a_1 \neq 0. \tag{33}$$

Substituting (33) into (31), setting coefficients of $(\frac{G'}{G})^i (i = 0, 1, ..., 4)$ to zero, we obtain the following under-determined system of algebraic equations for a_0, a_1, c, λ and μ:

$$\begin{aligned}
(\frac{G'}{G})^0 \quad &: \quad 2\beta {a_0}^4 + 2\alpha {a_0}^3 + 2c^2 a_0 a_1 \lambda\mu - 2c^2 {a_1}^2 \mu^2 + 2k{a_1}^2\mu^2 - 2ka_0 a_1 \lambda\mu = 0, \\
&\quad\; +2k{a_1}^2\mu^2 = 0, \\
(\frac{G'}{G})^1 \quad &: \quad 8\beta a_1 {a_0}^3 + 6\alpha a_1 {a_0}^2 + 2c^2 a_0 a_1 \lambda^2 + 4c^2 a_0 a_1 \mu - 2c^2 {a_1}^2 \lambda\mu - 2ka_0 a_1 \lambda^2 \\
&\quad\; -4ka_0 a_1 \mu + 2k{a_1}^2\lambda\mu = 0, \\
(\frac{G'}{G})^2 \quad &: \quad 12\beta {a_1}^2 {a_0}^2 + 6\alpha {a_1}^2 a_0 + 6c^2 a_0 a_1 \lambda - 6ka_0 a_1 \lambda = 0, \\
(\frac{G'}{G})^3 \quad &: \quad 2\alpha {a_1}^3 + 8\beta {a_1}^3 a_0 + 4c^2 a_0 a_1 + 2c^2 {a_1}^2 \lambda - 4ka_0 a_1 - 2k{a_1}^2 \lambda = 0, \\
(\frac{G'}{G})^4 \quad &: \quad 2\beta {a_1}^4 + 2c^2 {a_1}^2 - 2k{a_1}^2 = 0.
\end{aligned}$$

Solving this system by Maple, gives

- $$a_0 = -\frac{\alpha}{2\beta}(1 \mp \frac{\lambda}{\sqrt{\lambda^2 - 4\mu}}), \quad a_1 = \pm\frac{\alpha}{\beta\sqrt{\lambda^2 - 4\mu}}, \quad c = \pm\sqrt{k - \frac{\alpha^2}{\beta(\lambda^2 - 4\mu)}},$$
  $$k > c^2, \quad \beta > 0, \tag{34}$$

where λ and μ are arbitrary constants. Substituting Eq. (34) into Eq. (33) yields

$$v(\xi) = -\frac{\alpha}{2\beta}(1 \mp \frac{\lambda}{\sqrt{\lambda^2-4\mu}}) \pm \frac{\alpha}{\beta\sqrt{\lambda^2-4\mu}}(\frac{G'}{G}), \tag{35}$$

where $\xi = x - (\pm\sqrt{k - \frac{\alpha^2}{\beta(\lambda^2-4\mu)}})t$.
Substituting general solutions of Eq. (5) into Eq. (35), we have two types of traveling wave solutions of the double combined sinh-cosh-Gordon equation as follows:
When $\lambda^2 - 4\mu > 0$,

$$v_1(\xi) = -\frac{\alpha}{2\beta}(1 \mp \frac{c_1 \sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + c_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{c_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + c_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}), \tag{36}$$

from (30), we therefore obtain the solutions

$$\begin{aligned} u_1(\xi) = \ & \operatorname{arccosh}(\frac{1}{2}\{-\frac{\alpha}{2\beta}(1 \mp \frac{c_1 \sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + c_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{c_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + c_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}) \\ & -\frac{2\beta}{\alpha}(1 \mp \frac{c_1 \sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + c_2\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{c_1\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + c_2\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\xi})^{-1}\}), \end{aligned} \tag{37}$$

where $\xi = x - (\pm\sqrt{k - \frac{\alpha^2}{\beta(\lambda^2-4\mu)}})t$.
When $\lambda^2 - 4\mu < 0$,

$$v_2(\xi) = -\frac{\alpha}{2\beta}(1 \pm i\frac{-c_1 \sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + c_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{c_1\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + c_2\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}), \tag{38}$$

from (30), we therefore obtain the solutions

$$\begin{aligned} u_2(\xi) = \ & \operatorname{arccosh}(\frac{1}{2}\{-\frac{\alpha}{2\beta}(1 \pm i\frac{-c_1 \sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + c_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{c_1\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + c_2\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}) \\ & -\frac{2\beta}{\alpha}(1 \pm i\frac{-c_1 \sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + c_2\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{c_1\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + c_2\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\xi})^{-1}\}), \end{aligned} \tag{39}$$

where $\xi = x - (\pm\sqrt{k - \frac{\alpha^2}{\beta(\lambda^2-4\mu)}})t$.

In solutions (37) and (39), c_1 and c_2 are left as free parameters.
In particular, if $c_1 \neq 0$ and $c_2 = 0$, then u_1 becomes

$$u_1(\xi) = \operatorname{arccosh}\{\frac{1}{2}(-\frac{\alpha}{2\beta}(1 \mp \tanh[\frac{\alpha}{2\sqrt{\beta(k-c^2)}}(x-ct)]) \\ -\frac{2\beta}{\alpha(1 \mp \tanh[\frac{\alpha}{2\sqrt{\beta(k-c^2)}}(x-ct)])})\}, \quad (40)$$

if $c_2 \neq 0$, $c_1 = 0$,

$$u_1(\xi) = \operatorname{arccosh}\{\frac{1}{2}(-\frac{\alpha}{2\beta}(1 \mp \coth[\frac{\alpha}{2\sqrt{\beta(k-c^2)}}(x-ct)]) \\ -\frac{2\beta}{\alpha(1 \mp \coth[\frac{\alpha}{2\sqrt{\beta(k-c^2)}}(x-ct)])})\}, \quad (41)$$

where $k > c^2$ and $\beta > 0$, which are the solitary solutions of the double combined sinh-cosh-Gordon equation.
The solution (40) and (41) are same Eq. (45) and Eq. (46) in [18] respectively. Therefore the solutions in [18] are only a special case of the our solutions.

5.Conclusions

In this paper, the $(\frac{G'}{G})$-expansion method is used to conduct an analytic study on the combined sinh-cosh-Gordon and the double combined sinh-cosh-Gordon equations. The exact traveling wave solutions obtained in this study are more general, and it is not difficult to arrive at some known analytic solutions for certain choices of the parameters c_1 and c_2. Comparing the proposed method with the methods used in [18], show that the $(\frac{G'}{G})$-expansion method is not only simple and straightforward, but also avoids tedious calculations.

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