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STRONGLY CONVERGENT GENERALIZED DIFFERENCE SEQUENCE SPACES DEFINED BY A MODULUS

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Abstract. We introduce the strongly generalized difference $V^{\lambda}[A,\Delta^{m},p,f]$-summable sequences with respect to a modulus function and give the relation between the spaces of strongly generalized difference $V^{\lambda}[A,\Delta^{m},p]$ −summable sequences and strongly generalized difference $V^{\lambda}[A,\Delta^{m},p,f]$ −summable sequences with respect to a modulus function when $A=(a_{ik})$ is an infinite matrix of complex numbers and $p=(p_i)$ is a sequence of positive real numbers. We also give natural relationship between strongly generalized difference $V^{\lambda}[A,\Delta^{m},p,f]$-summable sequences with respect to a modulus function and strongly generalized difference $S^{\lambda}[A,\Delta^{m}]$-statistical convergence.

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1. Introduction

Throughout the article w denotes the space of all sequences. The studies on difference sequence spaces was initiated by Kizmaz [11] . This idea was further generalized by Et and Colak [7], Et and Esi [8], Esi and Tripathy [6], Tripathy et al.[22] and many others. For more details one may refer to these references.

Let $m\in\mathbb{N}$ be fixed, then the operation

$$\Delta^{m}:w\rightarrow w$$

is defined by

$$\Delta x_k=x_k-x_{k+1}$$

and

$$\Delta^{m}x_k=\Delta\left(\Delta^{m-1}x_k\right),\ \ (m\geq 2)$$

for all $k\in\mathbb{N}$, where $\Delta^{m}x_k=\Delta^{m-1}x_k-\Delta^{m-1}x_{k+1}$, $\Delta^{0}x_k=x_k$, for all $k\in\mathbb{N}$.

The generalized difference operator $\Delta^m x_k$ has the following binomial representation:

$$\Delta^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} x_{k+i}.$$

The notion of modulus function was introduced by Nakano [19] and Ruckle [21] .We recall that a modulus f is a function from $[0, \infty)$ to $[0, \infty)$ such that

(i) $f(x) = 0$ if and only if $x = 0$,

(ii) $f(x + y) \leq f(x) + f(y)$,

(iii) f is increasing from the right at 0.

It is immediate from (ii) and (iv) that f is continuous on $(0, \infty]$. Also, from condition (ii), we have $f(nx) \leq nf(x)$ for all $n \in \mathbb{N}$ and so $n^{-1}f(x) \leq f\left(xn^{-1}\right)$ for all $n \in \mathbb{N}$. A modulus function may be bounded (for example, $f(x) = x(1+x)^{-1}$) or unbounded (for example, $f(x) = x$). Ruckle [21] , Maddox [16] , Esi [5] and several authors used a modulus f to construct some sequence spaces.

Let $\Lambda = (\lambda_r)$ be a non-decreasing sequence of positive real numbers tending to infinity and $\lambda_1 = 1$ and $\lambda_{r+1} \leq \lambda_r + 1$. The generalized de la Vallee-Poussin means is defined by $t_r(x) = \lambda_r^{-1} \sum_{i \in I_r} x_i$, where $I_r = [r - \lambda_r + 1, r]$. A sequence $x = (x_i)$ is said to be (V, λ)-summable to a number L if $t_r(x) \to L$ as $r \to \infty$ (see for instance, Leindler [13]). If $\lambda_r = r$, then the (V, λ)-summability is reduced to ordinary $(C, 1)$-summability. A sequence $x = (x_i)$ is said to be strongly (V, λ)-summable to a number L if $t_r(|x - L|) \to 0$ as $r \to \infty$.

Let $A = (a_{ik})$ be an infinite matrix of complex numbers. We write $Ax = (A_i(x))_{i=1}^{\infty}$ if $A_i(x) = \sum_{k=1}^{\infty} a_{ik} x_k$ converges for each $i \in \mathbb{N}$.

Spaces of strongly summable sequences were discussed by Kuttner [12] , Maddox [14] and others. The class of sequences which are strongly Cesaro summable with respect to a modulus was introduced by Maddox [6] as an extension of the definition of strongly Cesaro summable sequences. Connor [2] further extended this definition to a definition of strongly A-summability with respect to a modulus when A is non-negative regular matrix.

Recently, the concept of strong (V, λ)-summability was generalized by Bilgin and Altun [1] as follows:

$$V^{\lambda}[A, p, f] = \left\{ x = (x_k) \in w : \lim_{r \to \infty} \lambda_r^{-1} \sum_{i \in I_r} \left[f\left(|A_i(x) - L| \right) \right]^{p_i} = 0, \text{ for some L} \right\}.$$

In the present paper we introduce the strongly generalized difference $V^{\lambda}[A, \Delta^m, p, f]$-summable sequences with respect to a modulus function and give the relation between the spaces of strongly generalized difference $V^{\lambda}[A, \Delta^m, p]$ $-$summable sequences and strongly generalized difference $V^{\lambda}[A, \Delta^m, p, f]$ $-$summable sequences with respect to a modulus function when $A=(a_{ik})$ is an infinite matrix of complex numbers and $p=(p_i)$ is a sequence of positive real numbers. We also give natural relationship between strongly generalized difference $V^{\lambda}[A,\Delta^m,p,f]$-summable sequences with respect to a modulus function and strongly $S^{\lambda}(A,\Delta^m)-$statistical convergence.

The following well-known inequality will be used throughout this paper:

$$|a_k+b_k|^{p_k}\leq T\left(|a_k|^{p_k}+|b_k|^{p_k}\right) \tag{1}$$

where a_k and b_k are complex numbers, $T=\max\left(1,2^{H-1}\right)$ and $H=\sup_k p_k<\infty$ (one may refer to Maddox [15]).

2.Main results

Let $A=(a_{ik})$ is an infinite matrix of complex numbers and $p=(p_i)$ be a bounded sequence of positive real numbers such that $0<h=\inf_i p_i\leq p_i\leq \sup_i p_i=H<\infty$ and f be a modulus. We define

$$V_1^{\lambda}[A,\Delta^m,p,f]=\left\{x=(x_k)\in w:\ \lim_{r\to\infty}\lambda_r^{-1}\sum_{i\in I_r}\left[f\left(|A_i\left(\Delta^m x\right)-L|\right)\right]^{p_i}=0\right\},$$

$$V_0^{\lambda}[A,\Delta^m,p,f]=\left\{x=(x_k)\in w:\ \lim_{r\to\infty}\lambda_r^{-1}\sum_{i\in I_r}\left[f\left(|A_i\left(\Delta^m x\right)|\right)\right]^{p_i}=0\right\},$$

$$V_\infty^{\lambda}[A,\Delta^m,p,f]=\left\{x=(x_k)\in w:\ \sup_r\lambda_r^{-1}\sum_{i\in I_r}\left[f\left(|A_i\left(\Delta^m x\right)|\right)\right]^{p_i}<\infty\right\},$$

where $A_i(\Delta^m x)=\sum_{k=1}^{\infty}a_{ik}\Delta^m x_k$.

A sequence $x=(x_i)$ is said to be strongly generalized difference $V_1^{\lambda}[A,\Delta^m,p,f]$-convergent to a number L if there is a complex number L such that $x=(x_i)\in V_1^{\lambda}[A,\Delta^m,p,f]$. In this case we write $x\to L\left(V_1^{\lambda}[A,\Delta^m,p,f]\right)$.

Throughout the paper β will denote one of the notations 0,1 or ∞.

When $f(x)=x$, then we write the sequence spaces $V_\beta^{\lambda}[A,\Delta^m,p]$ in place of $V_\beta^{\lambda}[A,\Delta^m,p,f]$.

If $p_i=1$ for all $i\in\mathbb{N}$, $V_\beta^{\lambda}[A,\Delta^m,p,f]$ reduce to $V_\beta^{\lambda}[A,\Delta^m,f]$. If $p_i=1$ for all $i\in\mathbb{N}, m=0$ and $\lambda_r=r$, the sequence spaces $V_\beta^{\lambda}[A,\Delta^m,p,f]$ reduce to $w_\beta(f_A)$ which were defined and studied by Esi and Et [5]. If $m=0$, $V_\beta^{\lambda}[A,\Delta^m,p,f]$ reduce to $V_\beta^{\lambda}[A,p,f]$. The sequence spaces $V_\beta^{\lambda}[A,p,f]$ were defined and studied by Bilgin and Altun [1].

In this section we examine some topological properties of $V_{\beta}^{\lambda}[A,\Delta^m,p,f]$ spaces and investigate some inclusion relations between these spaces.

Theorem 2.1. *Let f be a modulus function. Then, $V_{\beta}^{\lambda}[A,\Delta^m,p,f]$ is a linear space over the complex field $\mathbb{C}$ for β =0,1 or ∞.*

Proof. We give the proof only for β =0. Since the proof is analogous for the spaces $V_1^{\lambda}[A,\Delta^m,p,f]$ and $V_{\infty}^{\lambda}[A,\Delta^m,p,f]$, we omit the details.

Let $x,y\in V_0^{\lambda}[A,\Delta^m,p,f]$ and $\alpha,\mu\in\mathbb{C}$. Then there exists integers T_α and T_μ such that $|\alpha|\leq T_\alpha$ and $|\mu|\leq T_\mu$. By using (1) and the properties of modulus f, we have

$$\lambda_r^{-1}\sum_{i\in I_r}\left[f\left(\left|\sum_{k=1}^{\infty}a_{ik}\left(\Delta^m\left(\alpha x_k+\mu y_k\right)\right)\right|\right)\right]^{p_i}\leq$$

$$\leq\lambda_r^{-1}\sum_{i\in I_r}\left[f\left(\left|\sum_{k=1}^{\infty}\alpha a_{ik}\Delta^m x_k+\sum_{k=1}^{\infty}\mu a_{ik}\Delta^m y_k\right|\right)\right]^{p_i}$$

$$\leq T\lambda_r^{-1}\sum_{i\in I_r}\left[T_\alpha f\left(\left|\sum_{k=1}^{\infty}a_{ik}\Delta^m x_k\right|\right)\right]^{p_i}+T\lambda_r^{-1}\sum_{i\in I_r}\left[T_\mu f\left(\left|\sum_{k=1}^{\infty}a_{ik}\Delta^m y_k\right|\right)\right]^{p_i}$$

$$\leq TT_\alpha^H\lambda_r^{-1}\sum_{i\in I_r}\left[f\left(\left|\sum_{k=1}^{\infty}a_{ik}\Delta^m x_k\right|\right)\right]^{p_i}+TT_\mu^H\lambda_r^{-1}\sum_{i\in I_r}\left[f\left(\left|\sum_{k=1}^{\infty}a_{ik}\Delta^m y_k\right|\right)\right]^{p_i}$$

$$\to 0 \text{ as } r\to\infty.$$

This proves that $V_0^{\lambda}[A,\Delta^m,p,f]$ is linear.

Theorem 2.2. *Let f be a modulus function. Then the inclusions*

$$V_0^{\lambda}[A,\Delta^m,p,f]\subset V_1^{\lambda}[A,\Delta^m,p,f]\subset V_{\infty}^{\lambda}[A,\Delta^m,p,f]$$

hold.

Proof. The inclusion $V_0^{\lambda}[A,\Delta^m,p,f]\subset V_1^{\lambda}[A,\Delta^m,p,f]$ is obvious. Now let $x\in V_1^{\lambda}[A,\Delta^m,p,f]$ such that $x\to L\left(V_1^{\lambda}[A,\Delta^m,p,f]\right)$. By using (1), we have

$$\sup_r\lambda_r^{-1}\sum_{i\in I_r}\left[f\left(\left|A_i\left(\Delta^m x\right)\right|\right)\right]^{p_i}=\sup_r\lambda_r^{-1}\sum_{i\in I_r}\left[f\left(\left|A_i\left(\Delta^m x\right)-L+L\right|\right)\right]^{p_i}$$

$$\leq T\sup_r\lambda_r^{-1}\sum_{i\in I_r}\left[f\left(\left|A_i\left(\Delta^m x\right)-L\right|\right)\right]^{p_i}+T\sup_r\lambda_r^{-1}\sum_{i\in I_r}\left[f\left(|L|\right)\right]^{p_i}$$

$$\leq T\sup_r\lambda_r^{-1}\sum_{i\in I_r}\left[f\left(\left|A_i\left(\Delta^m x\right)-L\right|\right)\right]^{p_i}+T\max\left\{f\left(|L|\right)^h,f\left(|L|\right)^H\right\}<\infty.$$

Hence $x \in V_{\infty}^{\lambda}[A, \Delta^m, p, f]$. This shows that the inclusion

$$V_1^{\lambda}[A, \Delta^m, p, f] \subset V_{\infty}^{\lambda}[A, \Delta^m, p, f]$$

holds. This completes the proof.

Theorem 2.3. *Let $p = (p_i) \in l_{\infty}$. Then $V_0^{\lambda}[A, \Delta^m, p, f]$ is a paranormed space with*

$$g(x) = \sup_r \left(\lambda_r^{-1} \sum_{i \in I_r} [f(|A_i(\Delta^m x)|)]^{p_i} \right)^{\frac{1}{M}}$$

where $M = max(1, \sup_i p_i)$.

Proof. Clearly $g(-x) = g(x)$. It is trivial that $\Delta^m x_k = 0$ for $x = 0$. Hence we get $g(0) = 0$. Since $\frac{p_i}{M} \leq 1$ and $M \geq 1$, using the Minkowski's inequality and definition of modulus f, for each r, we have

$$\left(\lambda_r^{-1} \sum_{i \in I_r} [f(|A_i(\Delta^m(x+y))|)]^{p_i} \right)^{\frac{1}{M}}$$

$$\leq \left(\lambda_r^{-1} \sum_{i \in I_r} [f(|A_i(\Delta^m x) + f(A_i(\Delta^m y))|)]^{p_i} \right)^{\frac{1}{M}}$$

$$\leq \left(\lambda_r^{-1} \sum_{i \in I_r} [f(|A_i(\Delta^m x)|)]^{p_i} \right)^{\frac{1}{M}} + \left(\lambda_r^{-1} \sum_{i \in I_r} [f(|A_i(\Delta^m y)|)]^{p_i} \right)^{\frac{1}{M}}.$$

Now it follows that g is subadditive. Finally, to check the continuity of multiplication, let us take any complex number α. By definition of modulus f, we have

$$g(\alpha x) = \sup_r \left(\lambda_r^{-1} \sum_{i \in I_r} [f(|A_i(\Delta^m \alpha x)|)]^{p_i} \right)^{\frac{1}{M}} \leq K^{\frac{H}{M}} g(x)$$

where $K = 1 + [|\alpha|]$ ($[|t|]$ denotes the integer part of t). Since f is modulus, we have $x \to 0$ implies $g(\alpha x) \to 0$. Similarly $x \to 0$ and $\alpha \to 0$ implies $g(\alpha x) \to 0$.Finally, we have x fixed and $\alpha \to 0$ implies $g(\alpha x) \to 0$. This completes the proof.

Now we give relation between strongly generalized difference $V_{\beta}^{\lambda}[A, \Delta^m, p]$ $-$convergence and strongly generalized difference $V_{\beta}^{\lambda}[A, \Delta^m, p, f]$ $-$convergence.

Theorem 2.4. *Let f be a modulus function. Then*

$$V_{\beta}^{\lambda}[A, \Delta^m, p] \subset V_{\beta}^{\lambda}[A, \Delta^m, p, f].$$

Proof. We consider only the case $\beta = 1$. Let $x \in V_1^{\lambda}[A, \Delta^m, p]$ and $\varepsilon > 0$. We can choose $0 < \delta < 1$ such that $f(t) < \varepsilon$ for every $t \in [0, \infty)$ with $0 \leq t \leq \delta$. Then, we can write

$$\lambda_r^{-1} \sum_{i \in I_r} [f(|A_i(\Delta^m x) - L|)]^{p_i}$$

$$= \lambda_r^{-1} \sum_{\substack{i \in I_r \\ |A_i(\Delta^m x) - L| \leq \delta}} [f(|A_i(\Delta^m x) - L|)]^{p_i} + \lambda_r^{-1} \sum_{\substack{i \in I_r \\ |A_i(\Delta^m x) - L| > \delta}} [f(|A_i(\Delta^m x) - L|)]^{p_i}$$

$$\leq \max\left\{f(\varepsilon)^h, f(\varepsilon)^H\right\} + \max\left\{1, \left(2f(1)\delta^{-1}\right)^H\right\} \lambda_r^{-1} \sum_{\substack{i \in I_r \\ |A_i(\Delta^m x) - L| > \delta}} (|A_i(\Delta^m x) - L|)^{p_i}.$$

Therefore $x \in V_1^{\lambda}[A, \Delta^m, p, f]$.

Theorem 2.5. *Let f be a modulus function. If $\lim_{t \to \infty} \frac{f(t)}{t} = \phi > 0$, then $V_\beta^{\lambda}[A, \Delta^m, p] = V_\beta^{\lambda}[A, \Delta^m, p, f]$.*

Proof. For any modulus function, the existence of positive limit given with $\phi > 0$ was introduced by Maddox [17]. Let $\phi > 0$ and $x \in V_\beta^{\lambda}[A, \Delta^m, p, f]$. Since $\phi > 0$, we have $f(t) \geq \phi t$ for all $t \in [0, \infty)$. From this inequality, it is easy to see that $x \in V_\beta^{\lambda}[A, \Delta^m, p]$. By using Theorem 2.4., the proof is completed.

In the Theorem 2.5., the condition $\phi > 0$ can not be omitted. For this consider the following simple example.

Example 2.1. Let $f(x) = \ln(1 + x)$. Then $\phi = 0$. Now define $a_{ik} = 1$ for $i = k$, zero otherwise, $p_i = 1$ for all $i \in \mathbb{N}$ and $\Delta^m x_k$ to be $\lambda_r - th$ term in I_r for every $r \geq 1$ and $x_i = 0$ otherwise. Then we have

$$\lambda_r^{-1} \sum_{i \in I_r} [f(|A_i(\Delta^m x)|)]^{p_i} = \lambda_r^{-1} \ln(1 + \lambda_r) \to 0 \text{ as } r \to \infty$$

and so $x \in V_0^{\lambda}[A, \Delta^m, p, f]$, but

$$\lambda_r^{-1} \sum_{i \in I_r} (|A_i(\Delta^m x)|)^{p_i} = \lambda_r^{-1} \lambda_r \to 1 \text{ as } r \to \infty$$

and so $x \notin V_0^{\lambda}[A, \Delta^m, p]$.

Theorem 2.6. *Let $0 < p_i \leq q_i$ for all $i \in \mathbb{N}$ and let $\left(\frac{q_i}{p_i}\right)$ be bounded. Then $V_\beta^{\lambda}[A, \Delta^m, q, f] \subset V_\beta^{\lambda}[A, \Delta^m, p, f]$.*

Proof. If we take $b_i = [f(|A_i(\Delta^m x)|)]^{p_i}$ for all $i \in \mathbb{N}$, then using the same techique of Theorem 2 of Nanda [20], it is easy to prove the theorem.

Corollary 2.7. *The following statements are valid:*
(a) *If* $0 < inf_i p_i \leq 1$ *for all* $i \in \mathbb{N}$, *then* $V_\beta^\lambda[A, \Delta^m, f] \subset V_\beta^\lambda[A, \Delta^m, p, f]$.
(b) If $1 \leq p_i \leq \sup_i p_i = H < \infty$ for all $i \in \mathbb{N}$, then $V_\beta^\lambda[A, \Delta^m, p, f] \subset V_\beta^\lambda[A, \Delta^m, f]$.

*Proof.***(a).** It follows from Theorem 2.6 with $q_i = 1$ for all $i \in \mathbb{N}$.
(b) It follows from Theorem 2.6. with $p_i = 1$ for all $i \in \mathbb{N}$.

Theorem 2.8. Let $m \geq 1$ be a fixed integer, then $V_\beta^\lambda\left[A, \Delta^{m-1}, p, f\right] \subset V_\beta^\lambda[A, \Delta^m, p, f]$.

Proof. The proof of the inclusions follows from the following inequality

$$\lambda_r^{-1}\sum_{i\in I_r}[f(|A_i(\Delta^m x)|)]^{p_i} \leq T\lambda_r^{-1}\sum_{i\in I_r}\left[f\left(\left|A_i\left(\Delta^{m-1}x\right)\right|\right)\right]^{p_i}$$

$$+T\lambda_r^{-1}\sum_{i\in I_r}[f(|A_i(\Delta^m x)|)]^{p_i}.$$

3.S$^\lambda(A, \Delta^m)$ −STATISTICAL CONVERGENCE

In this section, we introduce natural relationship between strongly generalized $V_1^\lambda[A, \Delta^m, p, f]$ −convergence and strongly generalized difference $S^\lambda(A, \Delta^m)$ −statistical convergence. In [10], Fast introduced the idea of statistical convergence. These idea was later studied by Connor [2], Maddox [16], Mursaleen [18], Et and Nuray [9] ,Esi [5], Savaş [23] and many others.

A complex number sequence $x = (x_i)$ is said to be statistically convergent to the number L if for every $\varepsilon > 0$, $\lim_{n\to\infty}\left|\frac{K(\varepsilon)}{n}\right| = 0$, where $|K(\varepsilon)|$ denotes the number of elements in the set $K(\varepsilon) = \{i \in \mathbb{N} : \ |x_i - L| \geq \varepsilon\}$.

A complex number sequence $x = (x_i)$ is said to be strongly generalized difference $S^\lambda(A, \Delta^m)$ −statistically convergent to the number L if for every $\varepsilon > 0$, $\lim_{r\to\infty}\lambda_r^{-1}|KA(\Delta^m, \varepsilon)| = 0$, where $|KA(\Delta^m, \varepsilon)|$ denotes the number of elements in the set $KA(\Delta^m, \varepsilon) = \{i \in I_r : \ |A_i(\Delta^m x) - L| \geq \varepsilon\}$. The set of all strongly generalized difference $S^\lambda(A, \Delta^m)$ −statistically convergent sequences is denoted by $S^\lambda(A, \Delta^m)$.

If $m = 0$, $S^\lambda(A, \Delta^m)$ reduce to $S^\lambda(A)$ which was defined and studied by Bilgin and Altun [1]. If A is identity matrix and $\lambda_r = r$, $S^\lambda(A, \Delta^m)$ reduce to $S^\lambda(\Delta^m)$

which was defined and studied by Et and Nuray [9] . If $m = 0$ and $\lambda_r = r$, $S^{\lambda}(A, \Delta^m)$ reduce to S_A which was defined and studied by Esi [3] . If $m = 0$, A is identity matrix and $\lambda_r = r$, strongly generalized difference $S^{\lambda}(A, \Delta^m)$-statistically convergent sequences reduce to ordinary statistical convergent sequences.

Now we give the relation between strongly generalized difference $S^{\lambda}(A, \Delta^m)$ −statistical convergence and strongly generalized difference $V_1^{\lambda}[A, \Delta^m, p, f]$ −convergence.

Theorem 3.1. *Let f be a modulus function. Then*

$$V_1^{\lambda}[A, \Delta^m, p, f] \subset S^{\lambda}(A, \Delta^m).$$

Proof. Let $x \in V_1^{\lambda}[A, \Delta^m, p, f]$. Then

$$\lambda_r^{-1} \sum_{i \in I_r} [f(|A_i(\Delta^m x - L)|)]^{p_i} \geq \lambda_r^{-1} \sum_{\substack{i \in I_r \\ |A_i(\Delta^m x) - L| > \delta}} [f(|A_i(\Delta^m x) - L|)]^{p_i}$$

$$\geq \lambda_r^{-1} \sum_{\substack{i \in I_r \\ |A_i(\Delta^m x) - L| > \delta}} [f(\varepsilon)]^{p_i} \geq \lambda_r^{-1} \sum_{\substack{i \in I_r \\ |A_i(\Delta^m x) - L| > \delta}} \min\left(f(\varepsilon)^h, f(\varepsilon)^H\right)$$

$$\geq \min\left(f(\varepsilon)^h, f(\varepsilon)^H\right) \lambda_r^{-1} |KA(\Delta^m, \varepsilon)|.$$

Hence $x \in S^{\lambda}(A, \Delta^m)$.

Theorem 3.2. *Let f be a bounded modulus function. Then $V_1^{\lambda}[A, \Delta^m, p, f] = S^{\lambda}(A, \Delta^m)$.*

Proof. By Theorem 3.1., it is sufficient to show that $V_1^{\lambda}[A, \Delta^m, p, f] \supset S^{\lambda}(A, \Delta^m)$. Let $x \in S^{\lambda}(A, \Delta^m)$. Since f is bounded, so there exists an integer $K > 0$ such that $f(|A_i(\Delta^m x) - L|) \leq K$. Then for a given $\varepsilon > 0$,we have

$$\lambda_r^{-1} \sum_{i \in I_r} [f(|A_i(\Delta^m x) - L|)]^{p_i}$$

$$= \lambda_r^{-1} \sum_{\substack{i \in I_r \\ |A_i(\Delta^m x) - L| \leq \delta}} [f(|A_i(\Delta^m x) - L|)]^{p_i} + \lambda_r^{-1} \sum_{\substack{i \in I_r \\ |A_i(\Delta^m x) - L| > \delta}} [f(|A_i(\Delta^m x) - L|)]^{p_i}$$

$$\leq \max\left(f(\varepsilon)^h, f(\varepsilon)^H\right) + K^H \lambda_r^{-1} |KA(\Delta^m, \varepsilon)|.$$

Taking the limit as $\varepsilon \to 0$ and $r \to \infty$, it follows that $x \in V_1^{\lambda}[A, \Delta^m, p, f]$. This completes the proof.

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