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FUZZY BOOLEAN ALGEBRAS AND LUKASIEWICZ LOGIC

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Abstract. In this paper, we analyze the more adequate tools to solve many current logical challenges in A I. When approaching the complex practical problems, the choice of solving strategy is crucial. As we will see, the method to be used depends on many factors, as the type of problem or its applications. Sometimes, a combination of more than one approach may be used. For instance, a compound is many times possible, and may be further refined. In particular, we describe the behaviour of different fuzzy logics relative to principles as Contradiction and Excluded Middle Laws. So, we see if the class of fuzzy sets possess the character of Boolean Algebras, and the very interesting case considering Lukasiewicz Logic.

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1. Fuzzy Logic. Some previous concepts

Formal Language, when the syntax is precisely given. *Mathematical Logic* is the study of the formal languages. Usually, it is called Classical Logic, being dychotomic, or bi-valuated, only either True or False. Then, the *Fuzzy Logic* will be a generalization of the Mathematical or Classical Logic. It deals with the problem of the ambiguity in Logic. Classical Logics soon showed insufficiencies for the problems of A I. For this reason, a more flexible tool is needed, allowing for a gradation of certainty, indicating different degrees of membership to a set, or fulfilment of a property or relationship, and so on. The introduction of concepts and methods of Fuzzy Logic, where the idea of sets, relations and so on, must be modified in the sense of covering adequately the indetermination or imprecision of the real world.

We define the *"world"* as *a complete and coherent description of how things are or how they could have been.* In the problems related with this *"real world",* which is only one of the *"possible worlds",* the Monotonic Logic seldom works. Such type of Logic is the classical in formal worlds, such as Mathematics. But it is necessary to provide our investigations with a mathematical construct that can express all the "grey tones", not the classical representation of real world as either black or white, either all or nothing, but as in the common and natural reasoning, through progressive gradation.

2.Fuzzy Inference Rules

Fuzzy Rules are linguistic *IF-THEN* constructions that have the general form

"IF A THEN B"

where A and B are propositions containing linguistic variables.

A is called the *premise, or antecedent,* and B is the consequent (or action) of the rule.

In effect, the use of linguistic variables and fuzzy IF-THEN rules exploits the tolerance for imprecision and uncertainty.

In this respect, Fuzzy Logic imitates the ability of the human mind to summarize data and focus on decision-relevant information.

Hereditary Principle For each fuzzy set, A, associated to a fuzzy predicate, fuzzy subset of B, every element with the property A will inherit the property B.

Schematically,

$$\begin{array}{l} A(x) \\ A \subset B \\ \hline B(x), \\ \forall x \in U,\ A, B \subset U \end{array}$$

Observe the inclusion between fuzzy sets A and B: $A \subset B$, means, once traslated to membership degree functions,

$$\mu_A(x) < \mu_B(x), \ \forall x \in U$$

Composition Rule For each $x \in U$ that verifies the property defining the fuzzy set A, and such that this element is related (by R) with other element y, that belongs to a different universe, U_2, it is possible to associate to this element the composition between A and R. That is,

$$\begin{array}{l} A(x) \\ R(x, y) \\ \hline \forall x \in U,\ \forall y \in V,\ A \subset U,\ R \subset U \times V \end{array}$$

Generalized Modus Ponens Rule

$$\frac{\begin{array}{l} A(x) \\ B(x) \rightarrow C(y) \end{array}}{\begin{array}{l} [A \circ (B \times C)](y), \\ \forall x \in U,\ \forall y \in V,\ A,\ B \subset U,\ C \subset V \end{array}}$$

being

$$\mu_{A \circ (B \times C)}(y) \equiv \max \left(\min \left\{ \mu_A(x),\ \mu_{B \times C}(x, y) \right\} \right)$$

but then, according to the definition of Cartesian product of fuzzy sets,

$$\mu_{A \circ (B \times C)}(y) \equiv \max_{x \in U} (\min \{ \mu_A(x),\ \min [\mu_B(x),\ \mu_C(y)] \})$$

Generalized Modus Tollens Rule

$$\frac{\begin{array}{l} D(y) \\ B(x) \rightarrow C(y) \end{array}}{\begin{array}{l} [D \circ (C \times B)](x), \\ \forall x \in U,\ \forall y \in V,\ B \subset U,\ C,\ D \subset V \end{array}}$$

where

$$\mu_{[D \circ (C \times B)]}(x) = \max_{y \in V} \left(\min \left\{ \mu_D(y),\ \mu_{C \times B}(y, x) \right\} \right)$$

and so,

$$\mu_{[D \circ (C \times B)]}(x) = \max_{y \in V} (\min \{ \mu_D(y),\ \min [\mu_B(x),\ \mu_C(y)] \})$$

Note. The basic *difference between Modus Ponens and Modus Tollens* is that whereas in the former case, exact inference is produced, in the latter we may infer generalizations from a set of events.

Abduction Inference Rule Abduction infers plausible causes from an effect; concretely, the more plausible causes. So,

$$\frac{\begin{array}{l} B(y) \\ A(x) \rightarrow B(x) \end{array}}{\begin{array}{l} A(y), \\ \forall x \in U,\ \forall y \in V \end{array}}$$

Hypothetical Syllogism

$$\begin{array}{l} A\,(x) \rightarrow B\,(y) \\ B\,(y) \rightarrow C\,(z) \\ \hline [(A \times B) \circ (B \times C)]\,(x,\ z)\,, \\ \forall x \in U,\ \forall y \in V, \forall z \in W,\ A \subset U,\ B \subset V,\ C \subset W \end{array}$$

3. Fuzzy Relations

The composition of fuzzy relations is defined by the so called *"max-min product",* introduced previously by two fuzzy relations acting on subsequent universes of discourse, U_1, U_2, U_3,

$$R_1\,(U_1, U_2) \circ R_2\,(U_2, U_3) = R_3\,(U_1, U_3)$$

where

$$R_3\,(U_1, U_3) = \left\{(x, z) \mid \mu_{R_1 \circ R_2}\,(x, z)\,, \forall x \in U_1, \forall z \in U_3\right\}$$

being

$$\begin{aligned} \mu_{R_1 \circ R_2}\,(x, z) &= \max\left\{\forall y \in U_2 : \min\left(\mu_{R_1}\,(x, y)\,, \mu_{R_2}\,(y, z)\right)\right\} = \\ &= \max_{y \in U_2}\left\{\min\left(\mu_{R_1}\,(x, y)\,, \mu_{R_2}\,(y, z)\right)\right\} \end{aligned}$$

For this reason, the composition ($\circ$) of fuzzy relations will be often denominated *"max-min matricial product"*. But note that it is very different from the usual product among matrices.

As a particular case of the previous definition for the composition of fuzzy relations, we can introduce the composition of a fuzzy set and a fuzzy relation.

The usual *properties* of the classical relations can be translated to fuzzy relations, but modified in the following sense

- R is *Reflexive, if* $R\,(x,\ x) = 1,\ \forall x \in C \subset U$.

  Each element would be totally related with itself, when R is reflexive.

- R is S*ymmetric, if* $R(x, y) = R(y, x), \forall (x, y) \in C \times C \subset U \times U.$

  The principal diagonal, Δ, acts as a mirror, in the associated matrix.

- R is *Transitive, not in the usual* way for relations or associated matrices, but when the following holds

$$R(x, z) \geq \max_{y \in U} (\min \{R(x, y), R(y, z)\}), \forall (x,y) \in C \times C \subset U \times U$$

All these mathematical methods can be very interesting in Fuzzy Logic and also in many branches of Artificial Intelligence.

Connectives and Fuzzy Sets

We may define the classical operations among crisp, or classical sets, generalizing to fuzzy versions. So, they may be characterized by its membership functions.

We have, for the *union* of fuzzy sets, defined by

$$\mu_{F \cup G}(x) = \max \{\mu_F(x), \mu_G(x)\}$$

and *intersection* of fuzzy sets, by

$$\mu_{F \cap G}(x) = \min \{\mu_F(x), \mu_G(x)\}$$

and also the *complement* of a fuzzy set, F,by

$$\mu_{c(F)}(x) = 1 - \mu_F(x)$$

The *strict inclusion* among two Fuzzy sets may be introduced by

$$F \subset G \Leftrightarrow \mu_F(x) < \mu_G(x)$$

And in its more general (non strict) version,

$$F \subseteq G \Leftrightarrow \mu_F(x) \leq \mu_G(x)$$

The *difference* among two fuzzy sets, F and G, is expressed by

$$\mu_{F\ -\ G}(x) = \mu_{F\ \cap\ c(G)}(x) = \min\left\{\mu_F(x)\,,\ \mu_{c(G)}(x)\right\}$$

Hence, the *symmetric difference* of both fuzzy sets, very useful in many applications, would be

$$\mu_{F\ \triangle\ G}(x) = \mu_{(F-G)\ \cup\ (G-F)}(x) = \mu_{(F\ \cup\ G)-(F\ \cap\ G)}(x) = \\ = \mu_{(F\ \cup\ G)\ \cap\ c\ (F\ \cap\ G)}(x)$$

With these equalities, we can express, alternatively,

$$\mu_{F\ \triangle\ G}(x) = \max\left\{\mu_{F-G}(x)\,,\ \mu_{(G-F)}(x)\right\} = \\ = \max\left\{\mu_{F\ \cap\ c(G)}(x)\,,\ \mu_{G\ \cap\ c(F)}(x)\right\} = \\ = \max\left\{\min\left\{\mu_F(x)\,,\ \mu_{c(G)}(x)\right\}, \min\left\{\mu_G(x)\,,\ \mu_{c(F)}(x)\right\}\right\}$$

Or also, in an equivalent way,

$$\mu_{F\ \triangle\ G}(x) = \mu_{(F\ \cup\ G)\ \cap\ c\ (F\ \cap\ G)}(x) = \min\left\{\mu_{F\ \cup\ G}\ (x)\,,\ \mu_{c(F\ \cap\ G)}(x)\right\} = \\ = \min\left\{\max\left[\mu_F\ (x)\,,\ \mu_G\ (x)\right], 1 - \mu_{F\ \cap\ G}(x)\right\} = \\ = \min\left(\max\left[\mu_F\ (x)\,,\ \mu_G\ (x)\right], 1 - \left[\min\left(\mu_F(x)\,,\ \mu_G(x)\right)\right]\right)$$

The origin the name symmetric difference is clear, because the roles of F and G can be swapped, without changing the previous result,

$$\mu_{F\ \triangle\ G}(x) = \mu_{(F-G)\ \cup\ (G-F)}(x) = \\ = \mu_{(G-F)\ \cup\ (F-G)}(x) = \mu_{G\ \triangle\ F}(x)$$

Implication on Fuzzy Sets

Let A and B be two fuzzy sets, not necessarily on the same universe of discourse. The implication between them is the relation $R_{A\to B}$ such that

$$A \to B \equiv A\ X\ B$$

where x is an outer matricial product using the logical operator AND, denoted by $\wedge$, in this way

$$\begin{pmatrix} a_1 \\ a_2 \\ ... \\ a_n \end{pmatrix} x \begin{pmatrix} b_1 & b_2 & ... & b_m \end{pmatrix} =$$

$$= \begin{pmatrix} a_1 \wedge b_1 & a_1 \wedge b_2 & a_1 \wedge b_3 & ... & a_1 \wedge b_m \\ a_2 \wedge b_1 & a_2 \wedge b_2 & a_2 \wedge b_3 & ... & a_1 \wedge b_m \\ ... & ... & ... & ... & ... \\ a_n \wedge b_1 & a_n \wedge b_2 & a_n \wedge b_3 & ... & a_1 \wedge b_m \end{pmatrix}$$

An example would be

$$A = \begin{pmatrix} 0 \\ 0 \\ 0.6 \\ 1 \end{pmatrix}$$

$$and$$

$$B = \begin{pmatrix} 0 & 0.2 & 1 \end{pmatrix}$$

which gives

$$A\ X\ B = \begin{pmatrix} 0 \wedge 0 & 0 \wedge 0.2 & 0 \wedge 1 \\ 0 \wedge 0 & 0 \wedge 0.2 & 0 \wedge 1 \\ 0.6 \wedge 0 & 0.6 \wedge 0.2 & 0.6 \wedge 1 \\ 1 \wedge 0 & 1 \wedge 0.2 & 1 \wedge 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0.2 & 0.6 \\ 0 & 0.2 & 1 \end{pmatrix}$$

To each Fuzzy Predicate, we can associate *a Fuzzy Set*, defined by such property, that is, composed by the elements of the universe of discourse such that totally or parcially verify such condition.

So, we can prove that the class of fuzzy sets, with the operations $\cup$, $\cap$ and c (with c the complement set operation), does not constitute a Boolean Algebra, because neither the Contradiction Law nor the Third Excluded Principle hold.

Turning to the first mentioned definitions, both proofs can be expressed easily, by counter-examples, in an algebraic or geometric way.

4. Middle Excluded law

In the first case, the algebraic proof consists in seeing how the equality

$$X \cup c(X) = U$$

can be violated.

Translating to the membership functions characterisation,

$$\mu_{X \ \cup \ c(X)} = \mu_U$$

we see that

$$\mu_{X \ \cup \ c(X)} (x) = \mu_U \ (x), \forall x \in U$$

is wrong, for some $x \in U$.

We know that

$$\mu_u(x) = 1, \forall x$$

because $x \in U$ necessarily (all our elements are into U).

In the first member, we have

$$\mu_{X \ \cup \ c(X)} = \left(\max \left\{\mu_X, \mu_{c(X)}\right\}\right)(x) = (\max\{\mu_X, 1-\mu_X\})(x)$$

through the definition of the membership function for the union of fuzzy sets.

But if we take $x \mid \mu_X(x)$, such that $0 < \mu_X(x) < 1$, i. e. $\mu_X(x) \in (0,\ 1)$, the above equality does not holds.

For instance, if $\mu_X(x) = 0.2$, then $\mu_{c(X)}(x) = 0.8$, and so, we obtain

$$\mu_{X \ \cup \ c(X)} = \max \ \{0.2,\ 0.8\} = 0.8 \neq 1 = \mu_U(x)$$

This clearly fails. So, *the Third Excluded (or Middle Excluded) Law is not satisfied* in the family of Fuzzy Sets, if we take the precedent definitions.

Through a geometrical procedure, such proof can be showed by a very easy diagram.

For the *Contradiction Law*, in Fuzzy Sets, we must prove the possibility of

$$F \cap c(F) \neq \emptyset$$

or equivalently, that there exists $x \in U$ such that

$$\mu_{F\,\cap\,c(F)}(x) = \min\left(\mu_F(x), \mu_{c(F)}(x)\right) =$$
$$= \min\left(\mu_F(x), 1 - \mu_F(x)\right) \neq 0$$

As in the previous case, it is enough with taking x with membership degree between 0 and 1, both ends excluded.

For instance, if $\mu_F(x) = 0.3$, then $\mu_{c(F)}(x) = 0.7$. So,

$$\mu_{F\,\cap\,c(F)}(x) = \min\left(\mu_F(x), \mu_{c(F)}(x)\right) = \min(0.3, 0.7) = 0.3 \neq 0$$

Also here, through geometrical procedures, such proof can be showed by an easy diagram.

Therefore, we conclude that neither the Contradiction Law nor the Excluded Middle Law works in the class of Fuzzy Sets.

5. Into Lukasiewicz logic

But it is not so, if instead of these, we apply the Lukasiewicz alternative definitions.

It will be related with the *Lukasiewicz T-norm,*

$$T_{Luk}(a,\ b) \equiv \max\{0,\ a + b - 1\}$$

which is the standard semantics for strong conjunction in *Lukasiewicz fuzzy logic.*

And the *Lukasiewicz T-conorm,* also called *S-norm,* its dual, being the standard semantics for strong disjunction in *Lukasiewicz fuzzy logic,* given as

$$S_{Luk}(a,\ b) \equiv \min\{a + b,\ 1\}$$

Recall that given a T-norm, its complementary T-conorm, or *S-norm,* will be defined by the *De Morgan´s Laws,*

$$S_{Luk}(a,\ b) \equiv 1 - T_{Luk}(1 - a,\ 1 - b)$$

So, we have for the union of two fuzzy sets, A and B, included into the universe of discourse U, according Lukasiewicz,

$$A \cup B = \min\{1,\ A + B\} \Leftrightarrow \mu_{A\ \cup\ B} \equiv \min\left\{\mu_U,\ \mu_{A \cup B}\right\}$$

And for the intersection of both fuzzy sets, also according Lukasiewicz,

$$A \cap B = \max\{0,\ A + B - 1\} \Leftrightarrow \mu_{A\ \cap\ B} \equiv \max\left\{\mu_\varnothing,\ \mu_A + \mu_B - \mu_U\right\}$$

So, it holds

$$\mu_{A \cup A^c} \equiv \min\left\{\mu_U,\ \mu_{A \cup A^c}\right\}$$

where A^c is the fuzzy set complementary of A.

Let $x \in U$ be such that $\mu_A(x) = 0.4$. Then, $\mu_{A^c}(x) = 0.6$. Therefore,

$$\mu_{A \cup A^c} \equiv \min\left\{\mu_U,\ \mu_{A \cup A^c}\right\} = \min\left\{\mu_U,\ \mu_A(x) + \mu_{A^c}(x)\right\} =$$
$$= \min\{1,\ 0.4 + 0.6\} = 1 = \mu_U(x)$$

So, it verifies the *Excluded Middle Law.*

In a similar way, we can proceed with the Contradiction Principle, because

$$A \cap A^c = \varnothing \Leftrightarrow \mu_{A \cap A^c}(x) = \mu_\varnothing(x),\ \forall x \in U$$

According to the definition of intersection given by Lukasiewicz, and supposing newly the precedent belonging degrees,

$$\mu_{A \cap A^c} \equiv \max\left\{\mu_\varnothing(x),\ \mu_A(x) + \mu_{A^c}(x) - \mu_U(x)\right\} =$$
$$= \max\left\{0,\ \mu_A(x) + \mu_{A^c}(x)\right\} =$$
$$= \min\{1,\ 0.4 + 0.6 - 1\} = 0 = \mu_\varnothing(x)$$

Therefore, it also verifies the *Contradiction Law.*

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