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SUBORDINATION PROPERTIES FOR CERTAIN CLASS OF ANALYTIC FUNCTIONS DEFINED BY AN EXTENDED MULTIPLIER TRANSFORMATION

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Abstract. In this paper we derive several subordination results for certain class of analytic functions defined by an extended multiplier transformation.

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1. Introduction

Let A denote the class of functions of the form:

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k$$

which are analytic in the open unit disc $U=\{z:|z|<1\}$. We also denote by K the class of functions $f(z)\in A$ that are convex in U.

Many essentially equivalent definitions of multiplier transformation have been given in literature(see [5], [6], and [10]). In [4] Catas defined the operator $I^m(\lambda,\ell)$ as follows:

Definition 1 . *[4] Let the function $f(z)\in A$. For $m\in\mathbb{N}_o=\mathbb{N}\cup\{0\}$, where $\mathbb{N}=\{1,2,3,...\}$, $\lambda\geq 0$, $\ell\geq 0$. The extended multiplier transformation $I^m(\lambda,\ell)$ on A is defined by the following infinite series:*

$$I^m(\lambda,\ell)f(z)=z+\sum_{k=2}^{\infty}\left[\frac{1+\lambda(k-1)+\ell}{1+\ell}\right]^m a_kz^k. \quad (1)$$

It follows form(1.1) that $I^o(\lambda,\ell)f(z)=f(z)$,

$$\lambda zI^m\left((\lambda,\ell)f(z)\right)' = (1+\ell)I^{m+1}(\lambda,\ell)f(z)-(1-\lambda+\ell)I^m(\lambda,\ell)f(z) \quad (\lambda>0) \quad (2)$$

and

$$I^{m_1}(\lambda,\ell)(I^{m_2}(\lambda,\ell))f(z))=I^{m_1+m_2}(\lambda,\ell)f(z)=I^{m_2}(\lambda,\ell)(I^{m_1}(\lambda,\ell)f(z)). \quad (3)$$

for all integers m_1 and m_2.

We note that:
(i) $\mathrm{I}^m(1,0)f(z) = D^m f(z)$ (see[8]);
(ii) $I^m(\lambda,0)f(z) = D^m_\lambda f(z)$ (see [1]);
(iii) $I^m(1,\ell)f(z) = I^m(\ell)f(z)$ (see [5] and [6]);
(iv) $I^m(1,1)f(z) = I^m f(z)$ (see[10])..
Also if $f(z) \in A$, then we can write

$$I^m(\lambda,\ell)f(z) = \left(f * \varphi^m_{\lambda,\ell}\right)(z),$$

where

$$\varphi^m_{\lambda,\ell}(z) = z + \sum_{k=2}^{\infty}\left[\frac{1+\lambda(k-1)+\ell}{1+\ell}\right]^m z^k. \tag{4}$$

Let $G^m(\lambda,\ell,\mu,b)$ denote the subclass of A consisting of functions $f(z)$ which satisfy:

$$\mathrm{Re}\left\{1+\frac{1}{b}\left[(1-\mu)\frac{I^m(\lambda,\ell)f(z)}{z}+\mu\left(I^m(\lambda,\ell)f(z)\right)'-1\right]\right\}>0 \tag{5}$$

or which satisfy the following inequality:

$$\left|\frac{(1-\mu)\frac{I^m(\lambda,\ell)f(z)}{z}+\mu\left(I^m(\lambda,\ell)f(z)\right)'-1}{(1-\mu)\frac{I^m(\lambda,\ell)f(z)}{z}+\mu\left(I^m(\lambda,\ell)f(z)\right)'-1+2b}\right|<1 \tag{6}$$

where $z \in U; \mu \geq 0; \lambda > 0; \ell \geq 0; m \in \mathbb{N}_o; b \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$.
We note that:
(1)$G^m(1,0,\mu,b) = G_m(\mu,b)$ (see Aouf [2]);

$$=\left\{f\in A:\mathrm{Re}\left\{1+\frac{1}{b}\left[(1-\mu)\frac{D^m f(z)}{z}+\mu\left(D^m f(z)\right)'-1\right]\right\}>0;\ z\in U\right\} \tag{7}$$

(2) $G^m(\lambda,0,\mu,b) = G^m(\lambda,\mu,b)$

$$=\left\{f\in A:\mathrm{Re}\left\{1+\frac{1}{b}\left[(1-\mu)\frac{D^m_\lambda f(z)}{z}+\mu\left(D^m_\lambda f(z)\right)'-1\right]\right\}>0;\ z\in U\right\}; \tag{8}$$

(3) $G^m(1,\ell,\mu,b) = G^m(\ell,\mu,b)$

$$=\left\{f\in A:\mathrm{Re}\left\{1+\frac{1}{b}\left[(1-\mu)\frac{I^m(\ell)f(z)}{z}+\mu\left(I^m(\ell)f(z)\right)'-1\right]\right\}>0; z\in U\right\}; \tag{9}$$

(4) $G^m(1,1,\mu,b) = G^m(\mu,b)$

$$=\left\{f\in A:\mathrm{Re}\left\{1+\frac{1}{b}\left[(1-\mu)\frac{I^m f(z)}{z}+\mu\left(I^m f(z)\right)'-1\right]\right\}>0;\ z\in U\right\}; \tag{10}$$

(5) $G^m(\lambda, \ell, 0, b) = G^m(\lambda, \ell, b)$

$$= \left\{ f \in A : \operatorname{Re}\left[1 + \frac{1}{b}\left(\frac{I^m(\lambda, \ell)f(z)}{z} - 1\right)\right] > 0;\ z \in U \right\}; \tag{11}$$

(6) $\mathrm{G}^m(\lambda, \ell, 1, b) = R^m(\lambda, \ell, b)$

$$= \left\{ f \in A : \operatorname{Re}\left\{1 + \frac{1}{b}\left[(I^m(\lambda, \ell)f(z))' - 1\right]\right\} > 0;\ z \in U \right\}. \tag{12}$$

Definition 2 *(Hadamard Product or Convolution). Given two functions f and g in the class A, where $f(z)$ is given by (1.1) and $g(z)$ is given by*

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \tag{13}$$

the Hadamard product (or Convolution) $f * g$ is defined (as usual)by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z)\,(z \in U).$$

Definition 3 *(Subordination Principal). For two functions f and g, analytic in U, we say that the function $f(z)$ is subordinate to $g(z)$ in U, and write $f(z) \prec g(z)$ $(z \in U)$, if there exists a Schwarz function $\omega(z)$, which (by definition) is analytic in U with $\omega(0) = 0$ and $|\omega(z)| < 1$, such that $f(z) = g(\omega(z))\,(z \in U)$. Indeed it is known that*

$$f(z) \prec g(z)\,(z \in U) \Rightarrow f(0) = g(0) \textit{and } f(U) \subset g(U).$$

Furthermore, if the function g is univalent in U, then we have the following equivalence [7,p.4]

$$f(z) \prec g(z)\,(z \in U) \Leftrightarrow f(0) = g(0) \textit{and } f(U) \subset g(U).$$

Definition 4 *(Subordinatiry Factor Sequence). A sequence $\{b_k\}_{k=1}^{\infty}$of complex numbers is said to be a subordinating factor sequence if, wherever $f(z)$ of the form (1.1) is analytic, univalent and convex in U, we have the subordination given by*

$$\sum_{k=1}^{\infty} a_k b_k z^k \prec f(z)\ \ (z \in U; a_1 = 1). \tag{14}$$

2.main result

To prove our main result we need the following lemmas.

Lemma 1 *[11]. The sequence* $\{b_k\}_{k=1}^{\infty}$ *is a subordinating factor sequence if*

$$\operatorname{Re}\left\{1+2\sum_{k=1}^{\infty} b_k z^k\right\} > 0 \,(z\in U)\,.$$

Now, we prove the following lemma which gives a sufficient condition for functions belonging to the class $G^m(\lambda,\ell,\mu,b)$.

Lemma 2 *Let the function* $f(z)$ *which is defined by (1.1) satisfies the following condition:*

$$\sum_{k=2}^{\infty}[1+\mu(k-1)]\left[\frac{1+\lambda(k-1)+\ell}{1+\ell}\right]^m |a_k| \le |b| \qquad (\mu\ge 0;\lambda>0;\ell\ge 0;m\in\mathbb{N}_o;b\in\mathbb{C}^*)\,, \tag{15}$$

then $f(z)\in G^m(\lambda,\ell,\mu,b)$.

Proof. Suppose that the inequality (2.1) holds. Then we have for $z\in U$,

$$\left|(1-\mu)\frac{I^m(\lambda,\ell)f(z)}{z}+\mu\left(I^m(\lambda,\ell)f(z)\right)'-1\right|$$

$$-\left|(1-\mu)\frac{I^m(\lambda,\ell)f(z)}{z}+\mu\left(I^m(\lambda,\ell)f(z)\right)'+2b-1\right|$$

$$=\left|\sum_{k=2}^{\infty}[1+\mu(k-1)]\left[\frac{1+\lambda(k-1)+\ell}{1+\ell}\right]^m a_k z^{k-1}\right|$$

$$-\left|2b+\sum_{k=2}^{\infty}[1+\mu(k-1)]\left[\frac{1+\lambda(k-1)+\ell}{1+\ell}\right]^m a_k z^{k-1}\right|$$

$$\le\sum_{k=2}^{\infty}[1+\mu(k-1)]\left[\frac{1+\lambda(k-1)+\ell}{1+\ell}\right]^m |a_k|\,|z|^{k-1}$$

$$-\left\{2|b|-\sum_{k=2}^{\infty}[1+\mu(k-1)]\left[\frac{1+\lambda(k-1)+\ell}{1+\ell}\right]^m |a_k|\,|z|^{k-1}\right\}$$

$$\le 2\left\{\sum_{k=2}^{\infty}[1+\mu(k-1)]\left[\frac{1+\lambda(k-1)+\ell}{1+\ell}\right]^m |a_k|-|b|\right\}\le 0,$$

which shows that $f(z)$ belongs to the class $G^m(\lambda,\ell,\mu,b)$.

Let $G_*^m(\lambda,\ell,\mu,b)$ denote the class of functions $f(z)\in A$ whose coefficients satisfy the condition(2.1). We note that $G_*^m(\lambda,\ell,\mu,b)\subseteq G^m(\lambda,\ell,\mu,b)$.

Employing the technique used earlier by Attiya [3] and Srivastava and Attiya [9], we prove

Theorem 3 *Let $f(z)\in G_*^m(\lambda,\ell,\mu,b)$. Then*

$$\frac{(1+\mu)\left(\frac{1+\lambda+\ell}{1+\ell}\right)^m}{2\left[(1+\mu)\left(\frac{1+\lambda+\ell}{1+\ell}\right)^m+|b|\right]}\left(f*g\right)(z)\prec g(z)\ \ (z\in U) \tag{16}$$

for every function g in K, and

$$\operatorname{Re}\left(f(z)\right)>-\frac{\left[(1+\mu)\left(\frac{1+\lambda+\ell}{1+\ell}\right)^m+|b|\right]}{(1+\mu)\left(\frac{1+\lambda+\ell}{1+\ell}\right)^m}\left(z\in U\right). \tag{17}$$

The constant factor $\frac{(1+\mu)\left(\frac{1+\lambda+\ell}{1+\ell}\right)^m}{2\left[(1+\mu)\left(\frac{1+\lambda+\ell}{1+\ell}\right)^m+|b|\right]}$ in the subordination result (2.2) cannot be replaced by a larger one.

Proof. Let $f(z)\in G_*^m(\lambda,\ell,\mu,b)$ and let $g(z)=z+\sum_{k=2}^{\infty}c_kz^k\in K$. Then we have

$$\frac{(1+\mu)\left(\frac{1+\lambda+\ell}{1+\ell}\right)^m}{2\left[(1+\mu)\left(\frac{1+\lambda+\ell}{1+\ell}\right)^m+|b|\right]}\left(f*g\right)(z)=\frac{(1+\mu)\left(\frac{1+\lambda+\ell}{1+\ell}\right)^m}{2\left[(1+\mu)\left(\frac{1+\lambda+\ell}{1+\ell}\right)^m+|b|\right]}\left(z+\sum_{k=2}^{\infty}a_kc_kz^k\right). \tag{18}$$

Thus, by Definition 3, the subordination result (2.2) will hold true if the sequence

$$\left\{\frac{(1+\mu)\left(\frac{1+\lambda+\ell}{1+\ell}\right)^m}{2\left[(1+\mu)\left(\frac{1+\lambda+\ell}{1+\ell}\right)^m+|b|\right]}a_k\right\}_{k=1}^{\infty} \tag{19}$$

is a subordinating factor sequence with $a_1=1$.

In view of Lemma 1, this is equivalent to the following inequality:

$$\operatorname{Re}\left\{1+\sum_{k=1}^{\infty}\frac{(1+\mu)\left(\frac{1+\lambda+\ell}{1+\ell}\right)^m}{(1+\mu)\left(\frac{1+\lambda+\ell}{1+\ell}\right)^m+|b|}a_kz^k\right\}>0\ \ \left(z\in U\right). \tag{20}$$

Now, since

$$\Phi(k)=[1+\mu(k-1)]\left[\frac{1+\lambda(k-1)+\ell}{1+\ell}\right]^m$$

is an increasing function of $k (k\geq 2)$, we have

$$\mathrm{Re}\left\{1+\sum_{k=1}^{\infty}\frac{(1+\mu)\left(\frac{1+\lambda+\ell}{1+\ell}\right)^m}{(1+\mu)\left(\frac{1+\lambda+\ell}{1+\ell}\right)^m+|b|}a_kz^k\right\}=\mathrm{Re}\left\{1+\frac{(1+\mu)\left(\frac{1+\lambda+\ell}{1+\ell}\right)^m}{(1+\mu)\left(\frac{1+\lambda+\ell}{1+\ell}\right)^m+|b|}z\right.$$

$$\left.+\frac{1}{(1+\mu)\left(\frac{1+\lambda+\ell}{1+\ell}\right)^m+|b|}\sum_{k=2}^{\infty}(1+\mu)\left(\frac{1+\lambda+\ell}{1+\ell}\right)^m a_kz^k\right\}$$

$$\geq 1-\frac{(1+\mu)\left(\frac{1+\lambda+\ell}{1+\ell}\right)^m}{(1+\mu)\left(\frac{1+\lambda+\ell}{1+\ell}\right)^m+|b|}r$$

$$-\frac{1}{(1+\mu)\left(\frac{1+\lambda+\ell}{1+\ell}\right)^m+|b|}\sum_{k=2}^{\infty}[1+\mu(k-1)]\left[\frac{1+\lambda(k-1)+\ell}{1+\ell}\right]^m|a_k|r^k$$

$$>1-\frac{(1+\mu)\left(\frac{1+\lambda+\ell}{1+\ell}\right)^m}{(1+\mu)\left(\frac{1+\lambda+\ell}{1+\ell}\right)^m+|b|}r-\frac{|b|}{(1+\mu)\left(\frac{1+\lambda+\ell}{1+\ell}\right)^m+|b|}r=1-r>0\,(|z|=r<1)\,,$$

where we have also made use of the assertion (2.1) of Lemma 2. Thus (2.6) holds true in U. This proves the inequality (2.2). The inequality (2.3) follows from (2.2) by taking the convex function $g(z)=\frac{z}{1-z}=z+\sum_{k=2}^{\infty}z^k$. To prove the sharpness of the constant $\frac{(1+\mu)\left(\frac{1+\lambda+\ell}{1+\ell}\right)^m}{2\left[(1+\mu)\left(\frac{1+\lambda+\ell}{1+\ell}\right)^m+|b|\right]}$, we consider the function $f_o(z)\in G_*^m(\lambda,\ell,\mu,b)$ given by

$$f_o(z)=z-\frac{|b|}{(1+\mu)\left(\frac{1+\lambda+\ell}{1+\ell}\right)^m}z^2. \tag{21}$$

Thus from (2.2), we have

$$\frac{(1+\mu)\left(\frac{1+\lambda+\ell}{1+\ell}\right)^m}{2\left[(1+\mu)\left(\frac{1+\lambda+\ell}{1+\ell}\right)^m+|b|\right]}f_o(z)\prec\frac{z}{1-z}\,(z\in U)\,. \tag{22}$$

Morover, it can easily be verified for the function $f_o(z)$ given by (2.7) that

$$\min_{|z|\leq r}\left\{\operatorname{Re}\frac{(1+\mu)\left(\frac{1+\lambda+\ell}{1+\ell}\right)^m}{2\left[(1+\mu)\left(\frac{1+\lambda+\ell}{1+\ell}\right)^m+|b|\right]}f_o(z)\right\}=-\frac{1}{2}. \tag{23}$$

This shows that the constant $\frac{(1+\mu)\left(\frac{1+\lambda+\ell}{1+\ell}\right)^m}{2\left[(1+\mu)\left(\frac{1+\lambda+\ell}{1+\ell}\right)^m+|b|\right]}$ is the best possible.

Putting $\ell = 0$ in Theorem 1, we have

Corollary 4 *Let the function $f(z)$ defined by (1.1) be in the class $G_*^m(\lambda,\mu,b)$ and suppose that $g(z)\in K$. Then*

$$\frac{(1+\mu)(1+\lambda)^m}{2\left[(1+\mu)(1+\lambda)^m+|b|\right]}(f*g)(z)\prec g(z)\,(z\in U) \tag{24}$$

and

$$\operatorname{Re}(f(z))>-\frac{\left[(1+\mu)(1+\lambda)^m+|b|\right]}{(1+\mu)(1+\lambda)^m}\,(z\in U). \tag{25}$$

The constant factor $\frac{(1+\mu)(1+\lambda)^m}{2[(1+\mu)(1+\lambda)^m+|b|]}$ in the subordination result (2.10) cannot be replaced by a larger one.

Putting $\lambda = 1$ in Theorem 1, we have

Corollary 5 *Let the function $f(z)$ defined by (1.1) be in the class $G_*^m(\ell,\mu,b)$ and suppose that $g(z)\in K$.Then*

$$\frac{(1+\mu)\left(\frac{2+\ell}{1+\ell}\right)^m}{2\left[(1+\mu)\left(\frac{2+\ell}{1+\ell}\right)^m+|b|\right]}(f*g)(z)\prec g(z)\,(z\in U) \tag{26}$$

for every function g in K, and

$$\operatorname{Re}(f(z))>-\frac{\left[(1+\mu)\left(\frac{2+\ell}{1+\ell}\right)^m+|b|\right]}{(1+\mu)\left(\frac{2+\ell}{1+\ell}\right)^m}\,(z\in U). \tag{27}$$

The constsnt factor $\frac{(1+\mu)\left(\frac{2+\ell}{1+\ell}\right)^m}{2\left[(1+\mu)\left(\frac{2+\ell}{1+\ell}\right)^m+|b|\right]}$ in the subordination result (2.12) cannot be replaced by a larger one.

Putting $\lambda = \ell = 1$ in Theorem 1, we have

Corollary 6 *Let the function $f(z)$ defined by (1.1) be in the class $G_*^m(\mu, b)$ and suppose that $g(z) \in K$. Then*

$$\frac{(1+\mu)\left(\frac{3}{2}\right)^m}{2\left[(1+\mu)\left(\frac{3}{2}\right)^m + |b|\right]}\left(f * g\right)(z) \prec g(z)\,(z \in U) \tag{28}$$

for every function g in K and

$$\operatorname{Re}\left(f(z)\right) > -\frac{(1+\mu)\left(\frac{3}{2}\right)^m + |b|}{(1+\mu)\left(\frac{3}{2}\right)^m}\,(z \in U)\,. \tag{29}$$

The constant factor $\frac{(1+\mu)\left(\frac{3}{2}\right)^m}{2\left[(1+\mu)\left(\frac{3}{2}\right)^m+|b|\right]}$ in the subordination result (2.14) cannot be replaced by a larger one.

Putting $\mu = 0$ in Theorem 1, we have

Corollary 7 *Let the function $f(z)$ defined by (1.1) be in the class $G_*^m(\lambda, \ell, b)$ and suppose that $g(z) \in K$. Then*

$$\frac{\left(\frac{1+\lambda+\ell}{1+\ell}\right)^m}{2\left[\left(\frac{1+\lambda+\ell}{1+\ell}\right)^m + |b|\right]}\left(f * g\right)(z) \prec g(z)\,(z \in U) \tag{30}$$

for every function g in K, and

$$\operatorname{Re}\left(f(z)\right) > -\frac{\left[\left(\frac{1+\lambda+\ell}{1+\ell}\right)^m + |b|\right]}{\left(\frac{1+\lambda+\ell}{1+\ell}\right)^m}\,(z \in U)\,. \tag{31}$$

The constant factor $\frac{\left(\frac{1+\lambda+\ell}{1+\ell}\right)^m}{2\left[\left(\frac{1+\lambda+\ell}{1+\ell}\right)^m+|b|\right]}$ in the subordination result (2.16) cannot be replaced by a larger one.

Putting $\mu = 1$ in Theorem 1, we have

Corollary 8 *Let $f(z) \in R_*^m(\lambda, \ell, b)$. Then*

$$\frac{\left(\frac{1+\lambda+\ell}{1+\ell}\right)^m}{2\left(\frac{1+\lambda+\ell}{1+\ell}\right)^m + |b|}\left(f * g\right)(z) \prec g(z)\,(z \in U) \tag{32}$$

for every function g in K, and

$$\mathrm{Re}\left(f(z)\right) > -\frac{\left[2\left(\frac{1+\lambda+\ell}{1+\ell}\right)^m + |b|\right]}{2\left(\frac{1+\lambda+\ell}{1+\ell}\right)^m}\left(z \in U\right). \tag{33}$$

The constant factor $\frac{\left(\frac{1+\lambda+\ell}{1+\ell}\right)^m}{2\left(\frac{1+\lambda+\ell}{1+\ell}\right)^m+|b|}$ *in the subordination result (2.18) cannot be replaced by a larger one.*

Remark 1 *Putting* $\lambda = 1$ *and* $\ell = 0$ *in the above results we obtain the results obtained by Aouf [2].*

References

[1] F. M. Al-Oboudi, *On univalent functions defined by a generalized Salagean operator*, Internat. J. Math. Math. Sci. 27(2004), 1429-1436.

[2] M. K. Aouf, *Subordination properties for certain class of analytic functions defined by Salagean operator*, Applid. Math. Letters 22(2009), 1581-1585.

[3] A. A. Attiya, *On some application of a subordination theorems*, J. Math. Anal. Appl. 311(2005), 489-494.

[4] A. Catas, *A note on a certain subclass of analytic functions defined by multiplier transformations*, in Proceedings of the Internat. Symposium on Geometric function theory and Applications, Istanbul, Turkey, August 2007.

[5] N. E. Cho and H. M. Srivastava, *Argument estimates of certain analytic functions defined by a class of multiplier transformations*, Math. Comput. Modelling, 37(1-2)(2003), 39-49.

[6] N. E. Cho and T. H. Kim, *Multiplier transformations and strongly close-to-convex functions*, Bull. Korean Math. Soc. 40(2003), no. 3, 399-410.

[7] S. S. Miller and P. T. Mocanu, *Differentied subordinations: Theory and Applications*, Series on Monographs and Textbook in Pure and Appl. Math. no. 255 Marcel Dekker, Inc. New York, 2000.

[8] G. S. Salagean, *Subclasses of univalent functions*, Lecture Notes in Math.(Springer-Verlag), 1013(1983), 362-372.

[9] H. M. Srivastava and A. A. Attiya, *Some subordination rsults associated with certain subclass of analytic functions*, J. Inequal. Pure Appl. Math. 5(2004), Art. 82,1-6.

[10] B. A. Uralegaddi and C. Somanama, *Certain classes of univalent functions*, In: Current Topics in Analytic Function Theory, (Edited by H. M. Srivastava and S. Owa), 371-374, World Scientific, Publishing, Compeny, Singapore, 1992.

[11] H. S. Wilf, *Subordinating factor sequence for convex maps of the unit circle*, Proc. Amer. Math. Soc. 12(1961), 689-693.

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