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ON A SUBCLASS OF ANALYTIC FUNCTIONS DEFINED BY RUSCHEWEYH DERIVATIVE AND GENERALIZED SALAGEAN OPERATOR

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Abstract. Let $\mathcal{A}(p,n) = \{f \in \mathcal{H}(U): \ f(z) = z^p + \sum_{j=p+n}^{\infty} a_j z^j, \quad z \in U\}$, with $\mathcal{A}(1,1) = \mathcal{A}$. We consider in this paper the operator $RD_{\lambda,\gamma}^n : \mathcal{A} \to \mathcal{A}$, defined by $RD_{\lambda,\gamma}^n f(z) := (1-\gamma) R^n f(z) + \gamma D_\lambda^n f(z)$, where $D_\lambda^n f(z) = D_\lambda \left(D_\lambda^{n-1} f(z)\right)$ is the generalized Sălăgean operator and $(n+1)R^{n+1} f(z) = z(R^n f(z))' + nR^n f(z)$, $n \in \mathbb{N}_0$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ is the Ruscheweyh operator. By making use of the above mentioned differential operator, a new subclass of univalent functions in the open unit disc is introduced. The new subclass is denoted by $\mathcal{RD}^\lambda(n,\mu,\alpha,\lambda)$. Parallel results, for some related classes including the class of starlike and convex functions respectively, are also obtained.

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1. Introduction and definitions

Denote by U the unit disc of the complex plane, $U = \{z \in \mathbb{C}: \ |z| < 1\}$ and $\mathcal{H}(U)$ the space of holomorphic functions in U.

Let
$$\mathcal{A}(p,n) = \{f \in \mathcal{H}(U): \ f(z) = z^p + \sum_{j=p+n}^{\infty} a_j z^j, \quad z \in U\},$$
with $\mathcal{A}(1,n) = \mathcal{A}_n$, $\mathcal{A}(1,1) = \mathcal{A}_1 = \mathcal{A}$ and
$$\mathcal{H}[a,n] = \{f \in \mathcal{H}(U): \ f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots, \ z \in U\},$$
where $p, n \in \mathbb{N}$, $a \in \mathbb{C}$.

Let $\mathcal{S}$ denote the subclass of functions that are univalent in U.

By $\mathcal{S}^*(\alpha)$ we denote a subclass of $\mathcal{A}$ consisting of starlike univalent functions of order α, $0 \le \alpha < 1$ which satisfies
$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad z \in U. \tag{1}$$

Further, a function f belonging to $\mathcal{S}$ is said to be convex of order α in U, if and only if

$$\operatorname{Re}\left(\frac{zf''(z)}{f'(z)}+1\right)>\alpha, \quad z\in U, \tag{2}$$

for some α, $(0\leq\alpha<1)$. We denote by $\mathcal{K}(\alpha)$ the class of functions in $\mathcal{S}$ which are convex of order α in U and denote by $\mathcal{R}(\alpha)$ the class of functions in $\mathcal{A}$ which satisfy

$$\operatorname{Re} f'(z)>\alpha, \quad z\in U. \tag{3}$$

It is well known that $\mathcal{K}(\alpha)\subset\mathcal{S}^*(\alpha)\subset\mathcal{S}$.

If f and g are analytic functions in U, we say that f is subordinate to g, written $f\prec g$, if there is a function w analytic in U, with $w(0)=0$, $|w(z)|<1$, for all $z\in U$ such that $f(z)=g(w(z))$ for all $z\in U$. If g is univalent, then $f\prec g$ if and only if $f(0)=g(0)$ and $f(U)\subseteq g(U)$.

Definition 1. *(Al Oboudi [5]) For $f\in\mathcal{A}$, $\lambda\geq 0$ and $n\in\mathbb{N}$, the operator D_λ^n is defined by $D_\lambda^n:\mathcal{A}\to\mathcal{A}$,*

$$\begin{aligned}
D_\lambda^0 f(z) &= f(z)\\
D_\lambda^1 f(z) &= (1-\lambda)f(z)+\lambda z f'(z)=D_\lambda f(z)\\
&\dots\\
D_\lambda^{n+1} f(z) &= (1-\lambda)D_\lambda^n f(z)+\lambda z\left(D_\lambda^n f(z)\right)'=D_\lambda\left(D_\lambda^n f(z)\right), \text{ for } z\in U.
\end{aligned}$$

Remark 1. *If $f\in\mathcal{A}$ and $f(z)=z+\sum_{j=2}^{\infty}a_jz^j$, then*

$$D_\lambda^n f(z)=z+\sum_{j=2}^{\infty}\left[1+(j-1)\lambda\right]^n a_j z^j,$$

for $z\in U$.

Remark 2. *For $\lambda=1$ in the above definition we obtain the Sălăgean differential operator [10].*

Definition 2. *(Ruscheweyh [9]) For $f\in\mathcal{A}$ and $n\in\mathbb{N}$, the operator R^n is defined by $R^n:\mathcal{A}\to\mathcal{A}$,*

$$\begin{aligned}
R^0 f(z) &= f(z)\\
R^1 f(z) &= zf'(z)\\
&\dots\\
(n+1)R^{n+1} f(z) &= z\left(R^n f(z)\right)'+nR^n f(z), \quad \text{for } z\in U.
\end{aligned}$$

Remark 3. *If $f \in \mathcal{A}$ and $f(z) = z+\sum_{j=2}^{\infty} a_j z^j$, then $R^n f(z) = z+\sum_{j=2}^{\infty} C_{n+j-1}^{n} a_j z^j$, for $z \in U$.*

To prove our main theorem we shall need the following lemma.

Lemma 1. [8] *Let p be analytic in U with $p(0) = 1$ and suppose that*

$$\operatorname{Re}\left(1+\frac{zp'(z)}{p(z)}\right) > \frac{3\alpha-1}{2\alpha}, \quad z \in U. \tag{4}$$

Then $\operatorname{Re} p(z) > \alpha$ for $z \in U$ and $1/2 \leq \alpha < 1$.

2. Main results

Definition 3. *For a function $f \in \mathcal{A}$ we define the differential operator*

$$RD_{\lambda,\gamma}^{n} f(z) = (1-\gamma) R^n f(z) + \gamma D_{\lambda}^{n} f(z), \tag{5}$$

where $n \in \mathbb{N}_0$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Remark 4. *For $\lambda = 1$ the above defined operator was introduced in [1].*

Definition 4. *We say that a function $f \in \mathcal{A}$ is in the class $\mathcal{RD}^{\gamma}(n,\mu,\alpha,\lambda)$, $n \in \mathbb{N}$, $\mu \geq 0$, $\alpha \in [0,1)$, $\gamma \geq 0$ if*

$$\left|\frac{RD_{\lambda,\gamma}^{n+1} f(z)}{z}\left(\frac{z}{RD_{\lambda,\gamma}^{n} f(z)}\right)^{\mu} - 1\right| < 1-\alpha, \qquad z \in U. \tag{6}$$

Remark 5. *The family $\mathcal{RD}^{\gamma}(n,\mu,\alpha,\lambda)$ is a new comprehensive class of analytic functions which includes various new classes of analytic univalent functions as well as some very well-known ones. For example, $\mathcal{RD}^{1}(n,\mu,\alpha,\lambda)$ was studied in [6], $\mathcal{RD}^{0}(n,\mu,\alpha,\lambda)$ was studied in [3], $\mathcal{RD}^{\gamma}(n,\mu,\alpha,1)$ was studied in [4], $\mathcal{RD}^{1}(0,1,\alpha,1) = \mathcal{S}^{*}(\alpha)$, $\mathcal{RD}^{1}(1,1,\alpha,1) = \mathcal{K}(\alpha)$ and $\mathcal{RD}^{1}(0,0,\alpha,1) = \mathcal{R}(\alpha)$. Another interesting subclass is the special case $\mathcal{RD}^{1}(0,2,\alpha,1)=\mathcal{B}(\alpha)$ which has been introduced by Frasin and Darus [7] and also the class $\mathcal{RD}^{1}(0,\mu,\alpha,1) = \mathcal{B}(\mu,\alpha)$ which has been introduced by Frasin and Jahangiri [8].*

In this note we provide a sufficient condition for functions to be in the class $\mathcal{RD}^{\gamma}(n,\mu,\alpha,\lambda)$. Consequently, as a special case, we show that convex functions of order 1/2 are also members of the above defined family.

Theorem 2. *For the function $f\in\mathcal{A}$, $n\in\mathbb{N}$, $\mu\geq 0,\ 1/2\leq\alpha<1$ if*

$$(n+2)\frac{RD_{\lambda,\gamma}^{n+2}f(z)}{RD_{\lambda,\gamma}^{n+1}f(z)}-\mu(n+1)\frac{RD_{\lambda,\gamma}^{n+1}f(z)}{RD_{\lambda,\gamma}^{n}f(z)}-$$

$$\gamma\left(n+2-\frac{1}{\lambda}\right)\frac{D_{\lambda}^{n+2}f(z)-D_{\lambda}^{n+1}f(z)}{RD_{\lambda,\gamma}^{n+1}f(z)}+\mu\gamma\left(n+1-\frac{1}{\lambda}\right)\frac{D_{\lambda}^{n+1}f(z)-D_{\lambda}^{n}f(z)}{RD_{\lambda,\gamma}^{n}f(z)}$$

$$+(\mu-1)(n+1)\prec 1+\beta z,\quad z\in U, \tag{7}$$

where

$$\beta=\frac{3\alpha-1}{2\alpha},$$

then $f\in\mathcal{RD}^{\gamma}(n,\mu,\alpha,\lambda)$.

Proof. If we consider

$$p(z)=\frac{RD_{\lambda,\gamma}^{n+1}f(z)}{z}\left(\frac{z}{RD_{\lambda,\gamma}^{n}f(z)}\right)^{\mu}, \tag{8}$$

then $p(z)$ is analytic in U with $p(0)=1$. A simple differentiation yields

$$\frac{zp'(z)}{p(z)}=(n+2)\frac{RD_{\lambda,\gamma}^{n+2}f(z)}{RD_{\lambda,\gamma}^{n+1}f(z)}-\mu(n+1)\frac{RD_{\lambda,\gamma}^{n+1}f(z)}{RD_{\lambda,\gamma}^{n}f(z)}- \tag{9}$$

$$\gamma\left(n+2-\frac{1}{\lambda}\right)\frac{D_{\lambda}^{n+2}f(z)-D_{\lambda}^{n+1}f(z)}{RD_{\lambda,\gamma}^{n+1}f(z)}+\mu\gamma\left(n+1-\frac{1}{\lambda}\right)\frac{D_{\lambda}^{n+1}f(z)-D_{\lambda}^{n}f(z)}{RD_{\lambda,\gamma}^{n}f(z)}$$

$$+\mu(n+1)-(n+2).$$

Using (7) we get

$$\mathrm{Re}\left(1+\frac{zp'(z)}{p(z)}\right)>\frac{3\alpha-1}{2\alpha}.$$

Thus, from Lemma 1 we deduce that

$$\mathrm{Re}\left\{\frac{RD_{\lambda,\gamma}^{n+1}f(z)}{z}\left(\frac{z}{RD_{\lambda,\gamma}^{n}f(z)}\right)^{\mu}\right\}>\alpha.$$

Therefore, $f \in RD^{\gamma}(n, \mu, \alpha, \lambda)$, by Definition 4.

As a consequence of the above theorem we have the following interesting corollaries [2].

Corollary 3. *If $f \in \mathcal{A}$ and*

$$\operatorname{Re}\left\{\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}-\frac{zf''(z)}{f'(z)}\right\}>-\frac{1}{2}, \quad z \in U, \tag{10}$$

then

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\frac{1}{2}, \quad z \in U. \tag{11}$$

That is, f is convex of order $\frac{1}{2}$, or $f \in \mathcal{RD}^1\left(1,1,\frac{1}{2},1\right)$.

Corollary 4. *If $f \in \mathcal{A}$ and*

$$\operatorname{Re}\left\{\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)}\right\}>-\frac{1}{2}, \quad z \in U, \tag{12}$$

then $f \in \mathcal{RD}^1\left(1,0,\frac{1}{2},1\right)$, that is

$$\operatorname{Re}\left\{f'(z)+zf''(z)\right\}>\frac{1}{2}, \quad z \in U. \tag{13}$$

Corollary 5. *If $f \in \mathcal{A}$ and*

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\frac{1}{2}, \quad z \in U, \tag{14}$$

then

$$\operatorname{Re} f'(z)>\frac{1}{2}, \quad z \in U. \tag{15}$$

In another words, if the function f is convex of order $\frac{1}{2}$ then $f \in \mathcal{RD}^1(0,0,\frac{1}{2},1) \equiv \mathcal{R}\left(\frac{1}{2}\right)$.

Corollary 6. *If $f \in \mathcal{A}$ and*

$$\operatorname{Re}\left\{\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right\}>-\frac{3}{2}, \quad z \in U, \tag{16}$$

then f is starlike of order $\frac{1}{2}$, hence $f \in \mathcal{RD}^1(0,1,\frac{1}{2},1)$.

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