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CLASSES OF MEROMORPHIC FUNCTIONS WITH RESPECT TO N-SYMMETRIC POINTS

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Abstract. New subclasses of meromorphic functions with respect to N-symmetric points are defined and studied. Some properties of these classes are discussed. Moreover, subordination for meromorphic functions with respect to N-symmetric points is established.

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1. Introduction

Let $\mathcal{H}$ be the class of functions analytic in $U := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots$. Let $\mathcal{A}$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = z + a_2 z^2 + ...$ and $S = \{f \in \mathcal{A} : f \textit{ is univalent } \in U\}$. Let Σ be the class of meromorphic function in $\overline{U} := \{z \in \mathbb{C} : 0 < |z| < 1\}$, which takes the form

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n, \ \ (z \in \overline{U}).$$

Also let Θ be the class of the analytic functions in $\underline{U} := \{z \in \mathbb{C}_\infty : |z| > 1\}$, which takes the form

$$g(z) = z + a_0 + \sum_{n=1}^{\infty} \frac{a_n}{z^n}, \ \ (z \in \underline{U})$$

such that $g(\infty) = \infty$ and $g'(\infty) = 1$.

Definition 1.1. *A function $f(z) \in \Sigma$ is belongs to the class $\overline{S}^*$ if and only if*

$$\Re\{-\frac{zf'(z)}{f(z)}\} > 0, \quad (z \in \overline{U}).$$

And a function $g \in \Theta$ is belongs to the class $\underline{S}^*$ if and only if

$$\Re\{\frac{zg'(z)}{g(z)}\} > 0, \quad (z \in \underline{U}).$$

Note that some classes in Θ defined and established in [1,4].

Sakaguchi (see [6]) introduced the class of functions that are *starlike with respect to N-symmetric points*, $N = 1, 2, 3, \ldots$, as follows

$$\mathcal{SSP}_N = \left\{ f \in \mathcal{A} : \Re\{\frac{zf'(z)}{f_N(z)}\} > 0, z \in U \right\},$$

where

$$f_N(z) = z + \sum_{m=2}^{\infty} a_{m\cdot N+1} z^{m\cdot N+1}.$$

In order to give geometric characterization of the class $\mathcal{SSP}_N$ for $N \geq 2$ we define $\varepsilon := \exp(2\pi i/N)$ and we consider the weighted mean of $f \in \mathcal{A}$,

$$M_{f,N}(z) = \frac{1}{\sum_{j=1}^{N-1} \varepsilon^{-j}} \cdot \sum_{j=1}^{N-1} \varepsilon^{-j} \cdot f(\varepsilon^j z).$$

It is easy to verify that

$$\frac{f(z) - M_{f,N}(z)}{N} = \frac{1}{N} \cdot \sum_{j=0}^{N-1} \varepsilon^{-j} \cdot f(\varepsilon^j z) = f_N(z)$$

and further

$$f_N(\varepsilon^j z) = \varepsilon^j f_N(z),$$

$$f_N'(\varepsilon^j z) = f_N'(z) = \frac{1}{N} \sum_{j=0}^{N-1} f'(\varepsilon^j z),$$

$$\varepsilon^j f_N''(\varepsilon^j z) = f_N''(z) = \frac{1}{N} \sum_{j=0}^{N-1} \varepsilon^j f''(\varepsilon^j z).$$

Now, the class $\mathcal{SSP}_N$ is a collection of functions $f \in \mathcal{A}$ such that for any r close to 1, $r < 1$, the angular velocity of $f(z)$ about the point $M_{f,N}(z_0)$ is positive at $z = z_0$ as z traverses the circle $|z| = r$ in the positive direction. The authors received results concerning inclusion properties, some sufficient conditions for starlikeness with respect to N-symmetric points and sharp upper bound of the Fekete-Szegö functional $|a_3 - \mu a_2^2|$ over the classes $\mathcal{SSP}_N$ and $\mathcal{K}_N$ [7] where the class $\mathcal{K}_N$ of *convex functions with respect to N-symmetric points* is defined by

$$\Re\{\frac{[zf'(z)]'}{f_N'(z)}\} > 0, \quad z \in U.$$

For $N = 1$ we receive the well-known class of *starlike functions*, $S^* \equiv \mathcal{SSP}_1$, such that $f(U)$ is a starlike region with respect to the origin. And for $N = 2$ we receive $2f_2(z) = f(z) - f(-z)$, $M_{f,2}(z) = f(-z)$ and

$$\mathcal{SSP}_2 = \left\{ f \in \mathcal{A} : \Re\{\frac{zf'(z)}{f(z) - f(-z)}\} > 0, z \in U \right\}$$

is the class of *starlike functions with respect to symmetric points*.

Now we introduce classes of functions with N-symmetric points belong to the classes Σ and Θ as follows

$$\overline{S}_N^*(\alpha) := \{f \in \Sigma : \Re\{-\frac{zf'(z)}{f_N(z)}\} > \alpha, \;\; z \in \overline{U}, \;\; \alpha < 1\}$$

where $N = 1, 2, 3, ..$ and

$$f_N(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_{n.N+1} z^{n.N+1}, \;\; (z \in \overline{U}). \tag{1}$$

And the class

$$\underline{S}_N^*(\alpha) := \{g \in \Theta : \Re\{\frac{zg'(z)}{g_N(z)}\} > \alpha, \;\; z \in \underline{U}, \;\; 0 \le \alpha < 1\}$$

where $N = 1, 2, 3, ..$ and

$$g_N(z) = z + a_0 + \sum_{n=1}^{\infty} \frac{a_{n.N+1}}{z^{n.N+1}}, \;\; (z \in \underline{U}). \tag{2}$$

Note that when $N = 1$, the class $\underline{S}_N^*(\alpha)$ reduces to the class which investigated by Mocanu et [4] and Acu [1].

Moreover, we define the following subclasses

$$\overline{C}_N(\alpha) := \{f \in \Sigma : \Re\{-\frac{(zf'(z))'}{f'_N(z)}\} > \alpha, \ \ z \in \overline{U}, \ \alpha < 1\}.$$

And

$$\underline{C}_N(\alpha) := \{g \in \Theta : \Re\{\frac{(zg'(z))'}{g'_N(z)}\} > \alpha, \ \ z \in \underline{U}, \ 0 \leq \alpha < 1\}.$$

2. Some properties

In this section, we study some properties of the classes $\overline{S}^*_N$, $\underline{S}^*_N$, $\overline{C}_N$, and $\underline{C}_N$ in the following results.

Theorem 2.1. *Let* $f \in \Sigma$. *Then*

$$f(z) \in \overline{S}^*_N(\alpha) \ \ implies \ \ f_N(z) \in \overline{S}^*(\alpha),$$

where $\overline{S}^*(\alpha) := \overline{S}^*_1(\alpha)$ *is the class of starlike functions in* $\overline{U}$.

Proof. Suppose that $f(z) \in \overline{S}^*_N(\alpha)$, then from the definition we have

$$\Re\{-\frac{zf'(z)}{f_N(z)}\} > \alpha, \ \ (z \in \overline{U}).$$

Substituting z by $z\varepsilon^j$, where $\varepsilon^j = 1$, $j = 0, 1, ..., N-1$ yields

$$\begin{aligned}
&\Re\{-\frac{zf'(z)}{f_N(z)}\} > \alpha, \ \ (z \in \overline{U}) \Rightarrow \\
&\Re\{-\frac{z\varepsilon^j f'(z\varepsilon^j)}{f_N(z\varepsilon^j)}\} > \alpha, \ \ (z \in \overline{U}) \Rightarrow \\
&\Re\{-\frac{z\varepsilon^j f'(z\varepsilon^j)}{f_N(z)}\} > \alpha, \ \ (z \in \overline{U}) \Rightarrow \\
&\Re\{-\frac{z\sum_{j=0}^{N-1}\varepsilon^j f'(z\varepsilon^j)}{f_N(z)}\} > \alpha, \ \ (z \in \overline{U}) \Rightarrow \\
&\Re\{-\frac{zf'_N(z)}{f_N(z)}\} > \alpha, \ \ (z \in \overline{U}).
\end{aligned}$$

Hence $f_N(z) \in \overline{S}^*(\alpha)$.

By using the same method of Theorem 2.1, we have the following result.

Theorem 2.2. *Let* $g \in \Theta$. *Then*

$$g(z) \in \underline{S}^*_N(\alpha) \ \textit{implies} \ g_N(z) \in \underline{S}^*(\alpha),$$

where $\underline{S}^*(\alpha) := \underline{S}^*_1(\alpha)$ *is the class of starlike functions in* $\underline{U}$.

Theorem 2.3. *Let* $f \in \Sigma$. *Then*

$$f(z) \in \overline{C}_N(\alpha) \ \textit{implies} \ f_N(z) \in \overline{C}(\alpha),$$

where $\overline{C}(\alpha) := \overline{C}_1(\alpha)$ *is the class of convex functions in* $\overline{U}$.

Proof. Suppose that $f(z) \in \overline{C}_N(\alpha)$, then from the definition we have

$$\Re\{-\frac{(zf'(z))'}{f'_N(z)}\} > \alpha, \ \ (z \in \overline{U}).$$

Substituting z by $z\varepsilon^j$, where $\varepsilon^j = 1, \ j = 0, 1, ..., N-1$ yields

$$\begin{aligned}
&\Re\{-\frac{\left(f_N(z\varepsilon^j)\right)' + z\varepsilon^j\left(f_N(z\varepsilon^j)\right)''}{(f_N(z\varepsilon^j))'}\} > \alpha, \ \ (z \in \overline{U}) \Rightarrow \\
&\Re\{-\frac{\left(f_N(z\varepsilon^j)\right)' + z\varepsilon^j\left(f_N(z\varepsilon^j)\right)''}{(f_N(z))'}\} > \alpha, \ \ (z \in \overline{U}) \Rightarrow \\
&\Re\{-\frac{\sum_{j=0}^{N-1}\left(f_N(z\varepsilon^j)\right)' + z\sum_{j=0}^{N-1}\varepsilon^j\left(f_N(z\varepsilon^j)\right)''}{(f_N(z\varepsilon^j))'}\} > \alpha, \ \ (z \in \overline{U}) \Rightarrow \\
&\Re\{-\frac{z(f_N(z))'' + (f_N(z))'}{(f_N(z))'}\} > \alpha, \ \ (z \in \overline{U}) \Rightarrow \\
&\Re\{-\frac{(zf'_N(z))'}{f'_N(z)}\} > \alpha, \ \ (z \in \overline{U})
\end{aligned}$$

Hence $f_N(z) \in \overline{C}(\alpha)$.

By using the same method of Theorems 2.3, we have the following result.

Theorem 2.4. *Let* $g \in \Theta$. *Then*

$$g(z) \in \underline{C}_N(\alpha) \ \textit{implies} \ g_N(z) \in \underline{C}(\alpha).$$

where $\underline{C}(\alpha) := \underline{C}_1(\alpha)$ *is the class of convex functions in* $\underline{U}$.

Theorem 2.5. *Let* $f \in \Sigma$. *Then*

$$f(z) \in \overline{C}_N(\alpha) \ \ \Leftrightarrow \ \ zf'(z) \in \overline{S}^*_N(\alpha).$$

Proof. Let $h(z) = zf'(z)$. Then $h_N(z) = zf'_N(z)$ and

$$f \in \overline{C}_N(\alpha) \Leftrightarrow \Re\{-\frac{[zf'(z)]'}{f'_N(z)}\} > \alpha \Leftrightarrow \Re\{-\frac{zh'(z)}{h_N(z)}\} > \alpha \Leftrightarrow zf'(z) \in \overline{S}^*_N(\alpha).$$

In the same manner, we can receive the following result

Theorem 2.6. *Let $g \in \Theta$. Then*

$$g(z) \in \underline{C}_N(\alpha) \quad \Leftrightarrow \quad zg'(z) \in \underline{S}^*_N(\alpha).$$

3. Subordination results

Recall that the function F is *subordinate* to G, written $F \prec G$, if G is univalent, $F(0) = G(0)$ and $F(U) \subset G(U)$. In general, given two functions $F(z)$ and $G(z)$, which are analytic in U, the function $F(z)$ is said to be subordination to $G(z)$ in U if there exists a function $h(z)$, analytic in U with $h(0) = 0$ *and* $|h(z)| < 1$ *for all* $z \in U$ such that $F(z) = G(h(z))$ *for all* $z \in U$ (see[2,3]).

We need to the following result in sequel.

Lemma 3.1. [2] *Let $q(z)$ be univalent in the unit disk U and θ and ϕ be analytic in a domain D containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) := zq'(z)\phi(q(z))$, $h(z) := \theta(q(z)) + Q(z)$. Suppose that*

1. *$Q(z)$ is starlike univalent in U, and*
2. *$\Re\frac{zh'(z)}{Q(z)} > 0$ for $z \in U$.*

If $p(z)$ is analytic in U and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z))$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

Theorem 3.1. *Let $q(z)$ be univalent and $q(z) \neq 0$ in U and*

(1) *$\frac{zq'(z)}{q(z)}$ is starlike univalent in U, and*

(2) *$\Re\{1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} - \frac{q(z)}{\gamma}\} > 0$ for $z \in U$, $\gamma \neq 0$.*

If $f \in \Sigma$ and

$$-[(1-\gamma)\frac{zf'(z)}{f_N(z)} + \gamma\big(1 + \frac{zf''(z)}{f'_N(z)}\big)] \prec q(z) - \gamma\frac{zq'(z)}{q(z)},$$

then

$$-\frac{zf'(z)}{f_N(z)} \prec q(z) \tag{3}$$

and $q(z)$ is the best dominant.

Proof. Define the function $p(z)$ by

$$p(z) := -\frac{zf'(z)}{f_N(z)}.$$

A computation gives

$$p(z) - \gamma\frac{zp'(z)}{p(z)} = -[(1-\gamma)\frac{zf'(z)}{f_N(z)} + \gamma\big(1 + \frac{zf''(z)}{f'_N(z)}\big)]$$

where the functions θ and ϕ defined by

$$\theta(\omega) := \omega \quad \textit{and} \quad \phi(\omega) := -\frac{\gamma}{\omega}.$$

It can be easily observed that $\theta(\omega)$ is analytic in $\mathbb{C}$ and $\phi(\omega)$ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(\omega) \neq 0$ when $\omega \in \mathbb{C}\backslash\{0\}$. Also, by letting

$$Q(z) = zq'(z)\phi(q(z)) = -\gamma z\frac{q'(z)}{q(z)}$$

and

$$h(z) = \theta(q(z)) + Q(z) = q(z) - \gamma z\frac{q'(z)}{q(z)},$$

we find that $Q(z)$ is starlike univalent in U and that

$$\Re\{\frac{zh'(z)}{Q(z)}\} = \Re\{1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} - \frac{q(z)}{\gamma}\} > 0.$$

Then the relation (5) follows by an application of Lemma 3.1.

The next result can be found in [5].

Corollary 3.1. *Let the assumptions of Theorem 3.1 hold. Then*

$$-\frac{zf'(z)}{f(z)} \prec q(z) \tag{4}$$

and $q(z)$ is the best dominant.

Proof. By letting $N = 1$ in the above theorem the result follows immediately.

In Theorem 3.1, let

$$q(z) := \frac{1 + (1-2\alpha)z}{1-z},$$

we have the following result

Corollary 3.2. *Let $\alpha < 0, \gamma \neq 0$. If $f \in \Sigma$ and*

$$-[(1-\gamma)\frac{zf'(z)}{f_N(z)} + \gamma\big(1 + \frac{zf''(z)}{f'_N(z)}\big)] \prec \frac{1 + 2[1-\gamma+(\alpha-1)\gamma]z + (1-2\alpha)^2 z^2}{1-2\alpha z-(1-2\alpha)z^2},$$

then

$$-\Re\{\frac{zf'(z)}{f_N(z)}\} > \alpha. \tag{5}$$

Note that when $N = 1$, Corollary 3.2. reduces to result in [5, Corollary 3.1].

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References

[1] Acu M., *Some subclasses of meromorphic functions*, Filomat 21:2(2007), 1-9.

[2] Miller S.S., Mocanu P. T., *Differential Subordinations. Theory and Applications*, Marcel Dekker, New York-Basel, 2000.

[3] Miller S.S., Mocanu P.T., *Subordinants of differential superordinations*, Complex Variables, 48(10)(2003), 815-826.

[4] Mocanu P. T., Bulboaca T., Sălăgean G.S., *Teoria geometrica a functiilor univalente*, Casa Cartii de Stiinta(Cluj), 1999.

[5] Ravichandran V., Kumar S., Darus M., *On a subordination theorem for a class of meromorphic functions*, J. Ineq. Pure and Appli. Math., 5(8)(2004),1-4.

[6] Sakaguchi, K., *On a certain univalent mapping*, J. Math. Soc. Japan 11 (1959), 72-75.

[7] Tuneski N., Darus M., Ibrahim R.W., *Some results for univalent functions defined with respect to N-symmetric points*, submited.

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