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CUBICAL 2-COMPLEXES WITH THE 8-PROPERTY ADMIT A STRONGLY CONVEX METRIC

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Abstract. We study metric conditions implied by a combinatorial curvature condition on a finite cubical 2-complex. Our main result is that any finite, simply connected, cubical 2-complex with the 8-property, admits a strongly convex metric.

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1. Introduction

We study in this paper metric conditions implied by a combinatorial curvature condition, called the 8-property, on a finite, simply connected cubical 2-complex. Namely, we show that any finite, simply connected cubical 2-complex with the 8-property, admits a strongly convex metric.

Our proof uses results proven in [5] on square 2-complexes with the 8-property the fact that collapsible cubical 2-complexes admit a strongly convex metric. A proof of this fact is one of the paper's objects. W. White obtained in [8] a similar result. Namely, he showed that a collapsible simplicial 2-complex admits a strongly convex metric. Hence, since a finite, simply connected, simplicial 2-complex with the 6-property, collapses to a point (see [4]), we may immediately conclude that simplicial 2-complexes with the 6-property admit a strongly convex metric. We obtain in this paper a similar result on cubical 2-complexes with the 8-property.

In dimension 2, the 6-property (8-property) coincides with the CAT(0) property of the standard piecewise Euclidean metric on a simplicial (cubical) complex (see [3], chapter II.5, page 207). Therefore, since CAT(0) spaces have a strongly convex metric (see [3], chapter II.1, page 160), it is clear that simplicial (cubical) 2-complex with the 6-property (8-property) also have a strongly convex metric, when endowed with the standard piecewise Euclidean metric. Their collapsibility, however, implies that all simplicial (cubical) 2-complexes with the 6-property (8-property) admit a strongly convex metric, not only those endowed with the standard piecewise Euclidean metric.

2.Preliminaries

We present in this section basic facts about geometric notions such as distance, strongly convex metric, concave collection, elementary collapse and 8-property.

Definition 2.1. *Let* (X, d) *be a metric space. If* x, m, y *are three points in* X *such that* $d(x, m) + d(m, y) = d(x, y)$*, then we say that* m lies between x *and* y*. We call* m *the* midpoint *of* x *and* y *if* $d(x, m) = d(m, y) = \frac{1}{2}d(x, y)$.

Definition 2.2. *Let* (X, d) *be a metric space.* X *is a* convex *metric space if for any two points* x, y *in* X*, there exists at least one midpoint* m*.* X *is a* strongly convex *metric space if for any two points* x, y *in* X*, there exists exactly one midpoint* m*.*

Definition 2.3. *Let* (X, d) *be a metric space and let* x, y *be two distinct points in* X*. A* segment $c : [a, b] \to X$ *in* X *connecting* x *to* y *is a path which has, among all path joining* x *to* y *in* X*, the shortest length.*

Theorem 2.4. *Let* (X, d) *be a metric space. Let* x *and* y *be two distinct points in* X*.*

1. *A subset* S *of* X *containing* x *and* y *is a segment joining* x *to* y *if there exists a closed real line interval* $[a, b]$ *and an isometry* $c : [a, b] \to X$ *such that* $c(a) = x$ *and* $c(b) = y$*.*
2. *A path* $c : [a, b] \to X$ *joining* x *to* y *is a segment from* x *to* y *if and only if* $l(c) = d(x, y)$*.*

For the proof see [2], chapter II.2, page 76.

Theorem 2.5. *Let* (X, d) *be a complete metric space. There exists a segment in* X *(which is not necessarily unique) between any two distinct points* x, y *in* X *if and only if* X *is a convex metric space.*

For the proof we refer to [6].

Theorem 2.6. *Let* (X, d) *be a complete metric space. There exists a unique segment in* X *between any two distinct points* x, y *in* X *if and only if* X *is a strongly convex metric space.*

For the proof we refer to [6].

Definition 2.7. *Let* (X, d) *be a metric space. A* geodesic path *joining* $x \in X$ *to* $y \in X$ *is a path* $c : [a, b] \to X$ *such that* $c(a) = x$*,* $c(b) = y$ *and* $d(c(t), c(t')) = |t - t'|$ *for all* $t, t' \in [a, b]$*. The image* α *of* c *is called a* geodesic segment *with endpoints* x *and* y*. We denote any geodesic segment from a point* x *to a point* y *in* X*, by* $[x, y]$*.*

Definition 2.8. *Let X be a compact metric space with a strongly convex metric d. The* concave collection *T is a finite set of segments in X which satisfy the following condition: $\forall \rho, \tau \in T$, $\forall x_1, x_2 \in \rho$, $\forall y_1, y_2 \in \tau$, we have*

$$d(x_m, y_m) \leq \tfrac{1}{2}[d(x_1, y_1) + d(x_2, y_2)]$$

where we have denoted by x_m and y_m the midpoints of the segments $[x_1, x_2]$ and $[y_1, y_2]$.

The *unit n-cube* I^n is the n-fold product $[0,1]^n$; it is isometric to a cube in $\mathbb{R}^n$ with edges of length one. By convention, I^0 is a point. We will call a unit 2-cube simply a square.

We define a cubical complex by mimicking the definition of a simplicial complex, using unit cubes instead of simplices. Cubical complexes are more rigid objects than simplicial complexes and in many ways they are easier to work with.

Definition 2.9. *An n-dimensional* cubical complex *K is the quotient of a disjoint union of cubes $X = \bigcup_\Lambda I^{n_\lambda}$ by an equivalence relation $\sim$. The restrictions $p_\lambda : I^{n_\lambda} \to K$ of the natural projection $p : X \to K = X|_\sim$ are required to satisfy:*

1. *for every $\lambda \in \Lambda$, the map p_λ is injective;*
2. *if $p_\lambda(I^{n_\lambda}) \bigcap p_{\lambda'}(I^{n_{\lambda'}}) \neq \emptyset$, then there is an isometry $h_{\lambda,\lambda'}$ from a face $T_\lambda \subset I^{n_\lambda}$ onto a face $T_{\lambda'} \subset I^{n_{\lambda'}}$ such that $p_\lambda(x) = p_{\lambda'}(x')$ if and only if $x' = h_{\lambda,\lambda'}(x)$.*

In other words, K is a cubical complex if and only if each of its cells C_λ is isometric to a cube I^{n_λ}, each of the maps p_λ is injective, and the intersection of any two cells in K is empty or a single face.

There are many interesting examples of cubical complexes all of whose cells are cubes, but which do not satisfy all the conditions of the above definition (see [1], [3]). We use the term *cubed complex* to describe this larger class of complexes, except that in the 2-dimensional case we use the term *square complex*.

We define further the notion of collapsing a cell complex.

Definition 2.10. *Let K be a cell complex and let α be an i-cell of K. If β is a k-dimensional face of α but not of any other cell in K, then we say there is an* elementary collapse *from K to $K' = K \setminus \{\alpha, \beta\}$. We denote an elementary collapse by $K \searrow K'$. If $K = K_0 \supseteq K_1 \supseteq ... \supseteq K_n = L$ are cell complexes such that there is an elementary collapse from K_{j-1} to K_j, $1 \leq j \leq n$, then we say that K* collapses *to L.*

Let K be a cell complex. A *closed edge* is an edge together with its endpoints. An *oriented edge* of K is an oriented 1-cell of K, $e = [v_0, v_1]$. We denote by $i(e) = v_0$, the *initial* vertex of e, by $t(e) = v_1$, the *terminus* of e, and by $e^{-1} = [v_1, v_0]$, the *inverse* of e. A finite sequence $\alpha = e_1e_2...e_n$ of oriented closed edges in K such that $t(e_i) = i(e_{i+1})$ for all $1 \leq i \leq n-1$, is called an *edge-path* in K. If $t(e_n) = i(e_0)$, then we call α a *closed edge-path* or a *cycle*. We denote by $|\alpha|$ the number of 1-cells contained in α and we call $|\alpha|$ the *length* of α.

Definition 2.11. *Let K be a cell complex and let σ be a cell of K. The* link *of K at σ, denoted $Lk(\sigma, K)$, is the subcomplex of K consisting of all cells which are disjoint from σ and which together with σ span a cell of K.*

Definition 2.12. *Let K be a cell complex. A subcomplex L in K is called* full *(in K) if any cell of K spanned by a set of vertices in L is a cell of L. A* full cycle *in K is a cycle that is full as subcomplex of K.*

Definition 2.13. *Let K be a cell complex. We define the* systole *of K by*

$$sys(K) = \min\{|\alpha| : \alpha \text{ is a full cycle in } K\}.$$

Definition 2.14. *A 2- dimensional cell complex has the k-*property *if the link of each vertex is a graph of systole at least k, $k \in \{6, 8\}$.*

The main result of the paper is based on the following results.

Theorem 2.15. *Let K be a finite, simply connected, 2-dimensional square complex with the 8-property. Then K is collapsible.*

For the proof see [5].

Theorem 2.16. *Any finite cell complex that admits a strongly convex metric is contractible and locally contractible.*

For the proof we refer to [7].

3. Cubical 2-complexes with the 8-property admit a strongly convex metric

We prove in section that finite, simply connected cubical 2-complexes with the 8-property, admit a strongly convex metric. The result follows since finite, simply connected cubical 2-complexes with the 8-property, are collapsible (see [5]). An essential step in the proof is to show that collapsible cubical 2-complexes admit a strongly convex metric.

We start by proving an important lemma.

Lemma 3.17. *Let X be a finite metric space that admits a strongly convex metric d. Let T be a concave collection for d. Let σ be a 2-cell such that $X \cup \sigma$ is a metric space and $X \cap \sigma = \{\tau_1, \tau_2, \tau_3\}$ where τ_1, τ_2, τ_3 are segments contained in T. Let $abcd$ be a square and let $\varphi : abcd \to \sigma$ be a homeomorphism such that*

1. $\varphi(bc) = \tau_1$;
2. $\varphi(cd) = \tau_2$;
3. $\varphi(da) = \tau_3$;
4. $d(\varphi(x), \varphi(y)) = d_{\mathbb{R}^2}(x, y)$ *for all* $x, y \in \tau_i$, $1 \leq i \leq 3$.

Then there exists a strongly convex metric d' for $X \cup \sigma$ such that $T' = T \cup \{\varphi(ab)\}$ is a concave collection for d'.

Proof. Since $d(\varphi(x), \varphi(y)) = d_{\mathbb{R}^2}(x, y)$ for all $x, y \in \tau_i$, $1 \leq i \leq 3$, and $\varphi : abcd \to \sigma$ is a homeomorphism, we may attach the square $abcd$ to X along its sides bc, cd and da.

We define the metric d' as follows:

$$d'(x,y) = \begin{cases} d(x,y) \text{ for all } x, y \in X; \\ d_{\mathbb{R}^2}(\varphi^{-1}(x), \varphi^{-1}(y)) \text{ for all } x, y \in \sigma; \\ \min_{z \in \tau_1}\{d'(x,z) + d'(z,y)\} \text{ for all } x \in \sigma, y \in X \text{ or } x \in X, y \in \sigma; \\ \min_{z \in \tau_2}\{d'(x,z) + d'(z,y)\} \text{ for all } x \in \sigma, y \in X \text{ or } x \in X, y \in \sigma; \\ \min_{z \in \tau_3}\{d'(x,z) + d'(z,y)\} \text{ for all } x \in \sigma, y \in X \text{ or } x \in X, y \in \sigma. \end{cases}$$

We will prove that the metric d' defined above is a strongly convex metric by showing that any segment in $X \cup \sigma$ belongs to a concave collection $T' = T \cup \{\varphi(ab)\}$ for d'.

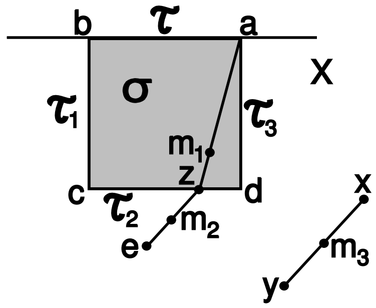

Figure 1.

Let $[a, e]$ be a segment in $X \cup \sigma$ and let $[x, y]$ be a segment in X. We denote by z the intersection point of $[a, e]$ and $[c, d]$. We denote by m_1 be the midpoint of $[a, e]$, by m_2 the midpoint of $[e, z]$, and by m_3 the midpoint of $[x, y]$.

We must show that the following relation holds

$$d'(m_1, m_3) \leq \frac{1}{2}[d'(a, x) + d'(e, y)]. \tag{1}$$

Firstly, we notice that

$$d'(m_1, m_2) = d'(e, m_1) - d'(e, m_2) = \frac{1}{2}d'(e, a) - \frac{1}{2}d'(e, z) = \frac{1}{2}d'(a, z). \tag{2}$$

Secondly, we notice that, since the segments $[z, e]$ and $[x, y]$ belong to the concave collection T for d, the following relation holds

$$d'(m_2, m_3) \leq \frac{1}{2}[d'(e, y) + d'(z, x)]. \tag{3}$$

Since z lies on the segment τ_2 which belongs to T, $d'(z, x) = d'(z, d) + d'(d, x)$. This holds since X is a complete strongly convex metric spaces. There exists therefore a unique segment $[d, x]$ ($[z, d]$) joining d to x (z to d) that belongs to T, and whose length equals $d'(d, x)$ ($d'(z, d)$). Similarly, since σ is a complete strongly convex metric spaces, there exists a unique segment $[a, z]$ joining a to z that belongs to σ, and whose length equals $d'(a, z)$.

Altogether, we have

$$\begin{aligned} & d'(m_1, m_3) \leq d'(m_1, m_2) + d'(m_2, m_3) \leq \\ & \leq \tfrac{1}{2}[d'(e, y) + d'(a, z) + d'(z, d) + d'(d, x)] = \\ & = \tfrac{1}{2}[d'(e, y) + d'(a, x)]. \end{aligned}$$

The above relation implies that the segment $[a, e]$ belongs to the concave collection T for d. The metric d' is therefore a strongly convex metric, and a concave collection T' for d' is $T' = T \cup \{\varphi(ab)\}$.

□

We present an important result of the section.

Theorem 3.18. *Let X be a 2-dimensional cubical complex that admits a strongly convex metric d. Let T be a concave collection for d which covers $|X^{(1)}|$. Let $\sigma^{(2)}$ and $\tau^{(1)}$ be two cells such that τ is a free face of the square σ. We consider the cubical 2-complex $X' = X \cup \{\sigma, \tau\}$ such that $X' \searrow X$ is an elementary collapse. Then X' admits a strongly convex metric d' such that $d'(x, y) = d(x, y)$ for all $x, y \in |X|$. Furthermore, there exists a concave collection T' for d' which covers $|X'^{(1)}|$.*

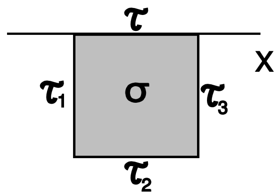

Figure 2.

Proof. Let $X \cap \sigma = \{\tau_1, \tau_2, \tau_3\}$. Because the concave collection T of d covers $|X^{(1)}|$, the segments τ_1, τ_2 and τ_3 belong to T. Hence, according to Lemma 3.18, $X' = X \cup \{\sigma, \tau\}$ admits a strongly convex metric d' and $T' = T \cup \tau$ is a concave collection for d' which covers $|X'^{(1)}|$. □

The above theorem implies the following result.

Corollary 3.19. *Any collapsible cubical* 2*-complex admits a strongly convex metric.*

According to Theorem 2.15, the above corollary implies the main result of the paper.

Corollary 3.20. *Any finite, simply connected cubical* 2*-complex with the* 8*-property, admits a strongly convex metric.*

According to Theorem 2.16, the following corollary holds.

Corollary 3.21. *Any finite, simply connected cubical* 2*-complex with the* 8*-property, is contractible and locally contractible.*

We note that, due to their collapsibility, it was already clear that finite, simply connected cubical 2-complexes with the 8-property are contractible.

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References

[1] M. Agoston, *Algebraic topology*, Marcel Dekker Inc., New York and Basel, 1976;

[2] A.-D. Aleksandrov, *Die innere Geometrie der konvexen Flaechen*, Akademie Verlag, Berlin, 1955;

[3] M. Bridson, A. Haefliger, *Metric spaces of non-positive curvature*, Springer, New York, 1999;

[4] J.-M. Corson, B. Trace, *The 6-property for simplicial complexes and a combinatorial Cartan-Hadamard theorem for manifolds*, Proceedings of the American Mathematical Society, 917-924, 1998;

[5] I.-C. Lazăr, *The collapsibility of square 2-complexes with the 8-property*, PhD Thesis, Proceedings of the 12th Symposium of Mathematics and its Applications, accepted;

[6] K. Menger, *Untersuchungen ueber allgemeine Metrik*, Math. Ann., 100, 1928, 75-163;

[7] D. Rolfsen, *Characterizing the 3-cell by its metric*, Fundamenta Mathematicae, LXVIII, 1970, 215-223;

[8] W. White, *A 2-complex is collapsible if and only if it admits a strongly convex metric*, Notices Amer. Math. Soc. 14, 848, 1967.

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