Acta Universitatis Apulensis No. 22/2010
ISSN: 1582-5329 pp. 41-45

A SHARP INEQUALITY INVOLVING THE PSI FUNCTION

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Abstract. The aim of this paper is to show that for $a \in (0,1)$, the function $f_a(x) = \psi(x+a) - \psi(x) - a/x$ is strictly completely monotonic in $(0,\infty)$. This result improves a previous result of Qiu and Vuorinen [Math. Comp. 74(2004) 723-742], who proved that $f_{1/2}$ is strictly decreasing and convex in $(0,\infty)$. As a direct consequence, a sharp inequality involving the psi function is established.

2000 *Mathematics Subject Classification*: 33B15, 05A16

1. Introduction and motivation

In the last decades, many authors have established various properties and bounds for special functions, as gamma or polygamma functions, *e.g.* [3-16]. For positive reals x, the Euler gamma function is defined as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt.$$

The gamma function was first introduced by the Swiss mathematician Leonhard Euler (1707-1783), when he was preoccupied to interpolate between the factorials $n!$, $n = 1, 2, 3, ...$. In this way, the gamma function is a natural generalization of the factorial function, since $\Gamma(n+1) = n!$, for every counting number n. The history and the development of this function are described in detail in [2, 3]. The logarithmic derivative of the gamma function, denoted

$$\psi(x) = \frac{d}{dx} \ln \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}$$

is called the Psi function, or digamma function. We use the fact that $\psi(1) = -\gamma$, where $\gamma = 0.57721566...$ is the Euler-Mascheroni constant and for every $x > 0$:

$$\psi(x+1) = \psi(x) + \frac{1}{x}. \tag{1.1}$$

The digamma function have the following asymptotic expansion

$$\psi(x) \sim \ln x - \frac{1}{2x} - \sum_{n=1}^{\infty} \frac{B_{2n}}{2nx^{2n}}, \tag{1.2}$$

where B_{2n} are the Bernoulli numbers defined by the relation

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n.$$

The derivatives ψ', ψ'', ... are known as the polygamma functions. They have the following integral representations:

$$\psi^{(n)}(x) = (-1)^{n+1} \int_0^{\infty} \frac{t^n}{1 - e^{-t}} e^{-tx} dt, \tag{1.3}$$

for $n = 1, 2, 3, \ldots$. See [1]. In what we are interested, we also use the following formulas, for $n \geq 1$,

$$\frac{1}{x^n} = \frac{1}{(n-1)!} \int_0^{\infty} t^{n-1} e^{-tx} dt. \tag{1.4}$$

Recall that a function h is (strictly) completely monotonic on $(0, \infty)$ if

$$(-1)^n f^{(n)}(x) \geq 0, \quad \text{respective} \quad (-1)^n f^{(n)}(x) > 0,$$

for every $x \in (0, \infty)$. The well-known Hausdorff-Bernstein-Widder theorem states that a function h is completely monotonic if and only if there exists a non-negative measure μ on $[0, \infty)$ such that for every $x \in (0, \infty)$,

$$h(x) = \int_0^{\infty} e^{-tx} d\mu(t).$$

For proofs and other details, see for example [1, 17]. Completely monotonic functions involving special functions are very important because they produce sharp bounds for the polygamma functions.

Recently, Qiu and Vuorinen obtained in [16, Theorem 2.1, p. 727] as an intermediary result, the monotonicity (strictly decreasing) and the convexity of the function

$$h_1(x) = \psi\left(x + \frac{1}{2}\right) - \psi(x) - \frac{1}{2x}$$

from $(0, \infty)$ onto $(0, \infty)$. Motivated by this result, we introduce, for every $a \in (0, 1)$, the class of functions $f_a : (0, \infty) \to (0, \infty)$, by the formula

$$f_a(x) = \psi(x + a) - \psi(x) - \frac{a}{x}$$

and we prove that f_a is strictly completely monotonic. As a direct consequence, f_a is strictly decreasing and convex on $(0, \infty)$. The particular case $a = 1/2$ is the result of Qiu an Vuorinen.

By using the complete monotonicity of the function $f_{1/2}$, we finally establish the following sharp inequality, for every $x \geq 1$,

$$0 < \psi\left(x + \frac{1}{2}\right) - \psi(x) \leq \omega,$$

where the constant $\omega = \frac{3}{2} - 2\ln 2 = 0.113\,71...$ is best possible. More generally, for every $a \in (0,1)$, we have

$$0 < \psi(x+a) - \psi(x) \leq \psi(a) + \gamma + \frac{1}{a} - a,$$

for every $x \geq 1$.

These estimations of the growth of the psi function are much used for studying the ratio of the gamma functions $\frac{\Gamma(x+a)}{\Gamma(x)}$, which has various applications in pure mathematics, as asymptotic expansions, refinements of the Wallis formula, Kazarinoff's inequality, or in applied mathematics, as probability theory, statistical physics, or mechanics.

2.The results

Now we are in position to prove the following main result:

Theorem 2.1. *For every $a \in (0,1)$, the function $f_a : (0, \infty) \to (0, \infty)$,*

$$f_a(x) = \psi(x+a) - \psi(x) - \frac{a}{x}$$

is strictly completely monotonic. In particular, f_a is strictly decreasing and convex.

Proof. We have

$$f_a'(x) = \psi'(x+a) - \psi'(x) + \frac{a}{x^2}$$

and using the integral representations (1.3)-(1.4), we obtain

$$f_a'(x) = \int_0^\infty \frac{t}{1-e^{-t}} e^{-(x+a)t} dt - \int_0^\infty \frac{t}{1-e^{-t}} e^{-xt} dt + \int_0^\infty a t e^{-tx} dt.$$

Straightforward computations lead us to the form

$$f_a'(x) = \int_0^\infty \frac{e^{-t(x+1)}}{1-e^{-t}} \varphi(t)\, dt,$$

where

$$\varphi(t) = te^{(1-a)t} - te^{t} + ate^{t} - at.$$

We have

$$\varphi(t) = \sum_{k=0}^{\infty} \frac{(1-a)^k}{k!} t^{k+1} - \sum_{k=0}^{\infty} \frac{1}{k!} t^{k+1} + \sum_{k=0}^{\infty} \frac{a}{k!} t^{k+1} - at,$$

or

$$\varphi(t) = \sum_{k=3}^{\infty} \frac{(1-a)\left[(1-a)^{k-2} - 1\right]}{(k-1)!} t^k < 0.$$

By the Hausdorff-Bernstein-Widder theorem, it results that $-f_a'$ is strictly completely monotonic. In particular, $f_a' < 0$, so f_a is strictly decreasing. As it results from (1.2), $\lim_{x\to\infty} f_a(x) = 0$, so $f_a > 0$. Finally, f_a is strictly completely monotonic.

As a direct consequence of the fact that f_a is strictly decreasing, we have for every $x \in [1, \infty)$,

$$0 = \lim_{x\to\infty} f_a(x) < f_a(x) \le f_a(1) = \psi(a+1) + \gamma - a,$$

or, using (1.1), we obtain

$$0 < \psi(x+a) - \psi(x) \le \psi(a) + \gamma + \frac{1}{a} - a$$

In particular, for $a = 1/2$, we obtain the following sharp inequality, for every $x \ge 1$,

$$0 < \psi\left(x + \frac{1}{2}\right) - \psi(x) \le \omega,$$

where the constant $f_{1/2}(1) = \omega = \frac{3}{2} - 2\ln 2 = 0.113\,71...$ is best possible.

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