SOME PROPERTIES OF THE SUBCLASS OF P-VALENT BAZILEVIC FUNCTIONS

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Abstract. In this paper, we define some new class, $B_k^\lambda(a,b,c,p,n,\alpha,\rho)$ by using the integral operators $I_{p,n}^\lambda(a,b,c)f(z)$. We also derive some interesting properties of functions belonging to the class $B_k^\lambda(a,b,c,p,n,\alpha,\rho)$. Our results generalizing the works of Owa and Cho, see [2, 10].

2000 *Mathematics Subject Classification*: 30C45, 30C50.

Keywords and phrases: Analytic functions; Bazilevic functions, Functions with positive real part, Integral operator.

1. Introduction

Let $\mathcal{A}_n(p)$ denote the class of functions $f(z)$ normalized by

$$f(z) = z^p + \sum_{k=n}^{\infty} a_{p+k} z^{p+k} \qquad (p, n \in N = \{1, 2, 3, ...\}) \tag{1.1}$$

which are analytic and p-valent in the unit disk

$$E = \{z \ : \ z \in \mathbb{C} \ and \ |z| < 1\}.$$

Let $P_k(\rho)$ be the class of functions $h(z)$ analytic in E satisfying the properties $h(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{Re\ h(z) - \rho}{1 - \rho} \right| d\theta \leq k\pi \tag{1.2}$$

where $z = re^{i\theta}, k \geq 2$ and $0 \leq \rho < 1$. This class has been introduced in [12]. We note, for $\rho = 0$ we obtain the class P_k defined and studied in [13], and for

$\rho = 0, k = 2$ we have the well known class P of functions with positive real part. The case $k = 2$ gives the class $P(\rho)$ of functions with positive real part greater than ρ. From (1.4) we can easily deduce that $h \in P_k(\rho)$ if, and only if, there exists $h_1, h_2 \in P(\rho)$ such that for $z \in E$,

$$h(z) = \left(\frac{k}{4} + \frac{1}{2}\right) h_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) h_2(z). \tag{1.3}$$

For functions $f_j(z) \in \mathcal{A}_n(p)$, given by

$$f_j(z) = z^p + \sum_{k=n}^{\infty} a_{p+k,j} z^{p+k} \qquad (j = 1, 2), \tag{1.4}$$

we define the Hadmard product (or convolution) of $f_1(z)$ and $f_2(z)$ by

$$(f_1 * f_2)(z) = z^p + \sum_{k=n}^{\infty} a_{p+k,1} a_{p+k,2} z^{p+k} = (f_2 * f_1)(z) \qquad (z \in E) \tag{1.5}$$

In our present investigation we shall make use of the Gauss hypergeometric functions defined by

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k z^k}{(c)_k (1)_k}, \tag{1.6}$$

where $a, b, c \in \mathbb{C}, a, b, c \notin \mathbb{Z}_0^- = \{0, -1, -2, ...\}$, and $(x)_k$ denote the Pochhammer symbol (or the shifted factorial) given, in terms of the Gamma function Γ, by

$$(x)_k = \frac{\Gamma(k+n)}{\Gamma(k)} = \begin{cases} x(x+1)...(x+k-1), & k \in N \\ 1, & k = 0 \end{cases}$$

We note that the series defined by (1.6) converges absolutely for $z \in E$ and hence ${}_2F_1(a, b; z)$ represents an analytic function in the open unit disk E, see [15].

We introduce a function $(z^p\ {}_2F_1(a, b; c; z))^{(-1)}$ given by

$$(z^p\ {}_2F_1(a, b; c; z))(z^p\ {}_2F_1(a, b; c; z))^{(-1)} = \frac{z^p}{(1-z)^{\lambda+p}}, \quad \lambda > -p, \tag{1.7}$$

this leads us to a family of linear operators:

$$I_{p,n}^{\lambda}(a, b, c)f(z) = (z^p\ {}_2F_1(a, b; c; z))^{(-1)} * f(z), \quad (a, b, c \in R \backslash Z_0^-, \lambda > -p, z \in E). \tag{1.8}$$

It is evident that $I^{\lambda}_{1,1}(a, n + 1, a)f(z)$ is the Noor integral operator, see [1, 4, 6, 7, 8, 9] which has fundament and significant applications in the geometric functions theory. The operator$I^{\lambda}_{p,n}(a, 1, c)$ was introduced by Cho et al [3].

After some computations, we obtain

$$I^{\lambda}_{p,n}(a, b, c)f(z) = z^p + \sum_{k=n}^{\infty} \frac{(c)_k(\lambda + p)_k}{(a)_k(b)_k} a_{p+k}z^{p+k} \tag{1.9}$$

From equation (1.8) we deduce that

$$I^{\lambda}_{p,n}(a, p + \lambda, a)f(z) = f(z) \textit{ and } I^{\lambda}_{p,n}(a, p, a) = \frac{zf'(z)}{p},$$

$$z(I^{\lambda}_{p,n}(a, b; c)f(z))' = (\lambda + p)I^{\lambda+1}_{p,n}(a, b; c)f(z) - \lambda I^{\lambda}_{p,n}(a, b; c)f(z), \tag{1.10}$$

Using the operator $I^{\lambda}_{p,n}(a, b; c)f(z)$ we now define a new subclass of $\mathcal{A}_n(p)$ as follows:

Definition 1.1. *Let $f(z) \in \mathcal{A}_n(p)$. Then $f(z) \in B^{\lambda}_k(a, b, c, p, n, \alpha, \rho)$ if and only if*

$$\left(\frac{I^{\lambda+1}_{p,n}(a, b; c)f(z)}{I^{\lambda}_{p,n}(a, b; c)f(z)}\right)\left(\frac{I^{\lambda}_{p,n}(a, b; c)f(z)}{z^p}\right)^{\alpha} \in P_k(\rho),$$

where $k \geq 2, \alpha > 0, 0 \leq \rho < p$ and $z \in E$.

Special Cases.

(i) $B^0_2(a, 1 + \lambda, a, 1, n, \alpha, \rho) = B(n, \alpha, \rho)$ is the subclass of Bazileivic functions studied by [2, 10].

(ii) The class $B^0_2(a, 1 + \lambda, a, 1, 1, \alpha, 0) = B(\alpha)$ is the subclass of Bazilevic functions which has been studied by Singh [14], see also [11].

2. Preliminaries and main results

Lemma 2.1. [5] *Let $u = u_1 + iu_2$ and $v = v_1 + iv_2$ and $\Psi(u, v)$ be a complex-valued function satisfying the conditions:*

(i) *$\Psi(u, v)$ is continuous in a domain $D \subset C^2$.*

(ii) *$(1, 0) \in D$ and $\Psi(1, 0) > 0$.*

(iii) *$Re\ \Psi(iu_2, v_1) \leq 0$ whenever $(iu_2, v_1) \in D$ and $v_1 \leq \frac{-n}{2}(1 + u_2^2)$.*

If $h(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \ldots$ *is analytic in* E *such that* $(h(z), zh'(z)) \in D$ *and* $Re\{\Psi(h(z), zh'(z))\} > 0$ *for* $z \in E$*, then* $Re\ h(z) > 0$ *in* E.

Theorem 2.2. *Let* $f(z) \in B_k^{\lambda}(a, b, c, p, n, \alpha, \rho)$*. Then*

$$\left\{\frac{I_{p,n}^{\lambda}(a,b,c)f(z)}{z^p}\right\}^{\alpha} \in P_k(\rho_1),$$

where ρ_1 *is given by*

$$\rho_1 = \frac{2\alpha(\lambda + p)\rho + np}{2\alpha(\lambda + p)\rho + n} \tag{2.1}$$

Proof. We begin by setting

$$\begin{aligned}\left\{\frac{I_{p,n}^{\lambda}(a,b,c)f(z)}{z^p}\right\}^{\alpha} &= (p - \rho_1)h(z) + \rho_1 \\ &= \left(\frac{k}{4} + \frac{1}{2}\right)((p - \rho_1)h_1(z) + \rho_1) \\ &- \left(\frac{k}{4} - \frac{1}{2}\right)((p - \rho_1)h_2(z) + \rho_1), \end{aligned} \tag{2.2}$$

so $h_i(z)$ is analytic in E, with $h_i(0) = 1, i = 1, 2$.

Taking logarithmic differentiation of (2.2), and using the identity (1.10) in the resulting equation we obtain

$$\left(\frac{I_{p,n}^{\lambda+1}(a,b,c)f(z)}{I_{p,n}^{\lambda}(a,b,c)f(z)}\right)\left(\frac{I_{p,n}^{\lambda}(a,b,c)f(z)}{z^p}\right)^{\alpha} =$$

$$= \left\{(p - \rho_1)h(z) + \rho_1 + \frac{(p - \rho_1)zh'(z)}{\alpha(\lambda + p)}\right\} \in P_k(\rho), \quad z \in E$$

This implies that

$$\frac{1}{p - \rho}\left\{(p - \rho_1)h_i(z) + \rho_1 - \rho + \frac{(p - \rho_1)zh'(z)}{\alpha(\lambda + p)}\right\} \in P, \quad z \in E, \quad i = 1, 2.$$

We form the functional $\Psi(u, v)$ by choosing $u = h_i(z), v = zh_i'(z)$.

$$\Psi(u, v) = \left\{(p - \rho_1)u + \rho_1 - \rho + \frac{(p - \rho_1)v}{\alpha(\lambda + p)}\right\}.$$

The first two conditions of Lemma 2.1 are clearly satisfied. We verify the condition (iii) as follows:

$$\begin{aligned} Re\{\Psi(iu_2, v_1)\} &= \rho_1 - \rho + Re\left\{\frac{(p-\rho_1)v_1}{\alpha(\lambda+p)}\right\} \\ &\leq \rho_1 - \rho - \frac{n(p-\rho_1)(1+u_2^2)}{2\alpha(\lambda+p)} \\ &= \frac{A+Bu_2^2}{2C}, \end{aligned}$$

where

$$A = 2\alpha(\lambda+p)(\rho_1-\rho) - n(p-\rho_1),$$

$$B = -n(p-\rho_1), \qquad C = 2\alpha(\lambda+p) > 0.$$

We notice that $Re\{\Psi(iu_2, v_1)\} \leq 0$ if and only if $A \leq 0, B \leq 0$ and this gives us ρ_1 as given by (2.1) and $B \leq 0$ gives us $0 \leq \rho_1 < 1$. Therefore applying Lemma 2.1, $h_i \in P, i = 1, 2$ and consequently $h \in P_k(\rho_1)$ for $z \in E$. This completes the proof.

Corollary 2.3. *If* $f(z) \in B_2^0(a, b, c, 1, n, \alpha, \rho)$, *then*

$$Re\left\{\frac{I_{p,n}^{\lambda}(a,b,c)f(z)}{z}\right\} > \frac{n+2\alpha\rho}{n+2\alpha}, \quad z \in E, \tag{2.3}$$

Corollary 2.4. *If* $f(z) \in B_2^0(a, b, c, 1, n, 1, 0)$, *then*

$$Re\left\{\frac{I_{p,n}^{\lambda}(a,b,c)f(z)}{z}\right\} > \frac{n}{n+2}, \quad z \in E, \tag{2.4}$$

Corollary 2.5. *If* $f(z) \in B_2^0(a, b, c, 1, n, \frac{1}{2}, \rho)$ *then*

$$Re\left\{\frac{I_{p,n}^{\lambda}(a,b,c)f(z)}{z}\right\} > \frac{\rho+n}{n+1}, \quad z \in E, \tag{2.5}$$

With certain choices of the parameters $\lambda, k, a, b, c, p, \alpha$ and ρ, we obtain the corresponding works of [2, 10].

Theorem 2.6. *Let* $f(z) \in B_k^\lambda(a,b,c,p,n,\alpha,\rho)$. *Then*

$$\left\{\frac{I_{p,n}^\lambda(a,b,c)f(z)}{z^p}\right\}^{\frac{\alpha}{2}} \in P_k(\lambda),$$

where

$$\lambda = \frac{np + \sqrt{(np)^2 + 4(\alpha(\lambda+p)+n)(\rho\alpha(\lambda+p))}}{2(\alpha(\lambda+p)+n)} \tag{2.6}$$

Proof. Let

$$\begin{aligned}\left\{\frac{I_{p,n}^\lambda(a,b,c)f(z)}{z^p}\right\}^{\alpha} &= ((p-\gamma)h(z)+\gamma)^2 \\ &= \left(\frac{k}{4}+\frac{1}{2}\right)((p-\gamma)h_1(z)+\gamma)^2 \\ &- \left(\frac{k}{4}-\frac{1}{2}\right)((p-\gamma)h_2(z)+\gamma)^2,\end{aligned} \tag{2.7}$$

so $h_i(z)$ is analytic in E with $h_i(0)=1,\ i=1,2.$

Taking logarithmic differentiation of (2.7), and using the identity (1.10) in the resulting equation we obtain that

$$\left(\frac{I_{p,n}^{\lambda+1}(a,b,c)f(z)}{I_{p,n}^\lambda(a,b,c)f(z)}\right)\left(\frac{I_p^\lambda(a,b,c)f(z)}{z^p}\right)^{\alpha} =$$

$$= \left[\{(p-\gamma)h(z)+\gamma\}^2 + \frac{2}{\alpha(\lambda+p)}\{(p-\gamma)h(z)+\gamma\}(p-\gamma)zh'(z)\right] \in P_k(\rho).$$

This implies that

$$\frac{1}{p-\rho}\left[\{(p-\gamma)h_i(z)+\gamma\}^2 + \frac{2}{\alpha(\lambda+p)}\{(p-\gamma)h_i(z)+\gamma\}(p-\gamma)zh_i'(z)-\rho\right] \in P,$$

where $z \in E, i = 1,2.$ We form the functional $\Psi(u,v)$ by choosing $u = h_i(z), v = zh_i'(z)$.

$$\Psi(u,v) = \{(p-\gamma)u+\gamma\}^2 + \left[\frac{2}{\alpha(\lambda+p)}\{(p-\gamma)u+\gamma\}(p-\gamma)v-\rho\right].$$

$$\begin{aligned} Re\{\Psi(iu_2, v_1)\} &= \gamma^2 - (p-\gamma)^2 u_2^2 + \left[\frac{2}{\alpha(\lambda+p)}\gamma(p-\gamma)v_1 - \rho\right] \\ &\leq \gamma^2 - \rho - (p-\gamma)^2 u_2^2 - \left[\frac{n\gamma(p-\gamma)(1+u_2^2)}{\alpha(\lambda+p)}\right] \\ &= \frac{A+Bu_2^2}{2C}, \end{aligned}$$

where

$$A = \gamma^2\alpha(\lambda+p) - \rho\alpha(\lambda+p) + n\gamma(p-\gamma),$$
$$B = -(p-\gamma)^2 - n\gamma(p-\gamma)$$
$$C = \frac{\alpha(\lambda+p)}{2} > 0.$$

We notice that $Re\{\Psi(iu_2, v_1)\} \leq 0$ if and only if $A \leq 0, B \leq 0$ and this gives us γ as given by (2.6) and $B \leq 0$ gives us $0 \leq \gamma < 1$. Therefore applying Lemma 2.1, $h_i \in P, i = 1, 2$ and consequently $h \in P_k(\rho_1)$ for $z \in E$. This completes the proof of Theorem 2.6.

Corollary 2.7. *If* $f(z) \in B_2^0(a, b, c, 1, n, \alpha, \rho)$ *then*

$$Re\left\{\frac{I_{p,n}^{\lambda}(a,b,c)f(z)}{z}\right\} > \frac{n + \sqrt{n^2 + 4(\alpha+n)(\rho\alpha)}}{2(\alpha+n)}, \quad z \in E.$$

Corollary 2.8. *If* $f(z) \in B_2^0(a, b, c, 1, n, 1, \rho)$ *then*

$$Re\left\{\frac{I_{p,n}^{\lambda}(a,b,c)f(z)}{z}\right\} > \frac{n + \sqrt{n^2 + 4(1+n)\rho}}{2(1+n)}, \quad z \in E.$$

Corollary 2.9. *If* $f(z) \in B_2^0(a, b, c, 1, n, 2, 0)$ *then*

$$Re\left\{\frac{I_{p,n}^{\lambda}(a,b,c)f(z)}{z}\right\} > \frac{n}{1+n}, \quad z \in E.$$

Again with certain choices of the parameters $\lambda, k, a, b, c, p, \alpha, \rho$ and we obtain the corresponding works of [2, 10].

Acknowledgement. We would like to thank Dr S. M. Junaid Zaidi, Rector CIIT for providing us excellent research facilities.

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