CRITERIA FOR UNIVALENCE OF CERTAIN INTEGRAL OPERATORS

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Abstract. In this paper, we determine conditions on β, α_i and $f_i(z)$ so that the integral operator $\left\{\beta\int_0^z \xi^{\beta-1}\prod_{i=1}^n \left(f_i(\xi)/\xi\right)^{1/\alpha_i} d\xi\right\}^{1/\beta}$ is univalent in the open unit disk. We also obtain similar results for the integral operator $\left\{\beta\int_0^z \xi^{\beta-1}\exp\left(\sum_{i=1}^n \alpha_i f_i(\xi)\right) d\xi\right\}^{\frac{1}{\beta}}$.

2000 *Mathematics Subject Classification*: 30C45.

1. Introduction

An analytic function that maps different points in the open unit disk $U := \{z \in \mathbb{C} : |z| < 1\}$ to different points in $\mathbb{C}$ is a univalent function. Such functions have been studied for long time. Let $\mathcal{A}$ be the class of all analytic functions $f(z)$ defined in U and normalized by the conditions $f(0) = 0 = f'(0) - 1$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions. Pascu [2] has proved the following theorem:

Theorem 1 *([1],[2]) Let $\beta \in \mathbb{C}$, $\gamma \in \mathbb{R}$ and $\Re\beta \geq \gamma > 0$. If $f \in \mathcal{A}$ satisfies*

$$\frac{1-|z|^{2\gamma}}{\gamma}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1 \quad (z \in U),$$

then the integral operator

$$F_\beta(z) = \left[\beta\int_0^z \xi^{\beta-1} f'(\xi) d\xi\right]^{\frac{1}{\beta}}$$

is in $\mathcal{S}$.

We denote by S_β the class of functions $f \in \mathcal{A}$ satisfying

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| < \beta \quad (0 < \beta \leq 1; \quad z \in U)$$

while S^*_β denote the class of functions $f \in \mathcal{A}$ satisfying

$$\left| \frac{z f'(z)}{f(z)} - 1 \right| < \beta \quad (0 < \beta \leq 1; \quad z \in U).$$

Using Theorem 1, Pescar [4] proved the following theorem.

Theorem 2 *([4]) Let $\alpha, \beta \in \mathbb{C}$ and $\Re\beta \geq \Re\alpha \geq 3/|\alpha|$. If $f \in S_1$ satisfies the condition*

$$|f(z)| \leq 1 \quad (z \in U),$$

then the integral operator

$$H_{\alpha,\beta}(z) = \left\{ \beta \int_0^z \xi^{\beta-1} \left(\frac{f(\xi)}{\xi} \right)^{\frac{1}{\alpha}} d\xi \right\}^{\frac{1}{\beta}} \tag{1}$$

is in $\mathcal{S}$.

Using Theorem 1, Breaz and Breaz [1] extended Theorem 2 and obtained the following theorem.

Theorem 3 *(Theorem 1, p.260 [1]) Let $\alpha, \beta \in \mathbb{C}$ and $\Re\beta \geq \Re\alpha > 3n/|\alpha|$. If $f_i \in S_1$ $(i = 1, 2, \cdots, n)$ satisfies the conditions*

$$|f_i(z)| \leq 1 \quad (z \in U, \quad i = 1, 2, \cdots, n),$$

then the integral operator

$$F_{\alpha,\beta}(z) = \left\{ \beta \int_0^z \xi^{\beta-1} \prod_{i=1}^{n} \left(\frac{f_i(\xi)}{\xi} \right)^{\frac{1}{\alpha}} d\xi \right\}^{\frac{1}{\beta}} \tag{2}$$

is in $\mathcal{S}$.

Using Theorem 1, Pescar [5] obtained the following theorem.

Theorem 4 *(Theorem 1, p.452 [5]) Let $f \in \mathcal{A}$ and $\beta \in \mathbb{C}$ satisfies*

$$1 \leq \Re\beta \leq |\beta| \leq \frac{3\sqrt{3}}{2}.$$

If

$$|zf'(z)| \leq 1 \quad (z \in U),$$

then the function

$$T_\beta(z) = \left\{ \beta \int_0^z \xi^{\beta-1} \left(e^{f(\xi)}\right)^\beta d\xi \right\}^{\frac{1}{\beta}}$$

is in $\mathcal{S}$.

By Schwarz's Lemma (see below), the only function $f \in \mathcal{A}$ with $|f(z)| \leq 1$ is $f(z) = z$ and hence the hypothesis of Theorem 3–4 are satisfied only by a single function $f(z) = z$.

In this paper, we extend Theorem 2 and Theorem 4 to obtain a sufficient conditions for univalence of a more general integral operator. We also prove some related results. The class of functions to which our theorems are applicable is non-trivial (when $M_i \neq 1$ for some i).

To prove our main results, we need the following lemma:

Lemma 1 (Schwarz's Lemma) *If the function $w(z)$ is analytic in the unit disk U, $w(0) = 0$ and $|w(z)| \leq 1$ for all $z \in U$, then*

$$|w(z)| \leq |z| \quad (z \in U), \quad |w'(0)| \leq 1$$

and the either of these equalities holds if and only if $w(z) = \epsilon z$, where $|\epsilon| = 1$.

2. Univalence Criteria

For $f_i \in \mathcal{A}$ $(i = 1, 2, \cdots, n)$ and $\alpha_1, \alpha_2, \ldots, \alpha_n, \beta \in \mathbb{C}$, we define an integral operator by

$$F_{\alpha_1,\alpha_2,\cdots,\alpha_n;\beta}(z) := \left\{ \beta \int_0^z \xi^{\beta-1} \prod_{i=1}^n \left(\frac{f_i(\xi)}{\xi} \right)^{\frac{1}{\alpha_i}} d\xi \right\}^{\frac{1}{\beta}}. \tag{3}$$

When $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha$, $F_{\alpha_1,\alpha_2,\cdots,\alpha_n;\beta}(z)$ becomes the integral operator $F_{\alpha,\beta}(z)$ considered in Theorem 3.

Theorem 5 *Let* $0 < \beta_i \leq 1$*. Let* $f_i \in S_{\beta_i}$ $(i = 1, 2, \cdots, n)$ *satisfy the conditions*

$$|f_i(z)| \leq M_i \quad (M_i \geq 1; \quad z \in U, \quad i = 1, 2, \cdots, n).$$

If $\alpha_1, \alpha_2, \ldots, \alpha_n, \beta \in \mathbb{C}$, $\Re\beta \geq \gamma$ *and*

$$\gamma := \sum_{i=1}^{n} \frac{1 + M_i(1 + \beta_i)}{|\alpha_i|}, \tag{4}$$

then the function $F_{\alpha_1,\alpha_2,\cdots,\alpha_n;\beta}(z)$ *defined by* (3) *is in* $\mathcal{S}$*.*

Proof. Define the function $h(z)$ by

$$h(z) := \int_0^z \prod_{i=1}^{n} \left(\frac{f_i(\xi)}{\xi}\right)^{\frac{1}{\alpha_i}} d\xi.$$

Then we have $h(0) = h'(0) - 1 = 0$. Also a simple computation yields

$$h'(z) = \prod_{i=1}^{n} \left(\frac{f_i(z)}{z}\right)^{\frac{1}{\alpha_i}}$$

and

$$\frac{zh''(z)}{h'(z)} = \sum_{i=1}^{n} \frac{1}{\alpha_i} \left(\frac{zf_i'(z)}{f_i(z)} - 1\right). \tag{5}$$

From equation (5), we have

$$\begin{aligned} \left|\frac{zh''(z)}{h'(z)}\right| &\leq \sum_{i=1}^{n} \frac{1}{|\alpha_i|} \left(\left|\frac{zf_i'(z)}{f_i(z)}\right| + 1\right) \\ &= \sum_{i=1}^{n} \frac{1}{|\alpha_i|} \left(\left|\frac{z^2 f_i'(z)}{f_i^2(z)}\right| \left|\frac{f_i(z)}{z}\right| + 1\right). \end{aligned} \tag{6}$$

From the hypothesis, we have $|f_i(z)| \leq M_i \quad (z \in U, \quad i = 1, 2, \cdots, n)$ which, by Schwartz Lemma, yields

$$|f_i(z)| \leq M_i|z| \quad (z \in U, \quad i = 1, 2, \cdots, n).$$

Using this inequality in the inequality (6), we obtain

$$\begin{aligned}\left|\frac{zh''(z)}{h'(z)}\right| &\leq \sum_{i=1}^{n}\frac{1}{|\alpha_i|}\left(M_i\left|\frac{z^2f_i'(z)}{f_i^2(z)}\right|+1\right)\\ &\leq \sum_{i=1}^{n}\frac{1}{|\alpha_i|}\left(M_i\left|\frac{z^2f_i'(z)}{f_i^2(z)}-1\right|+1+M_i\right). \qquad (7)\end{aligned}$$

Since $f_i \in S_{\beta_i}$, in view of (4), the equation (7) yields

$$\left|\frac{zh''(z)}{h'(z)}\right| < \sum_{i=1}^{n}\frac{1+M_i(1+\beta_i)}{|\alpha_i|} = \gamma. \qquad (8)$$

Multiply (8) by

$$\frac{1-|z|^{2\gamma}}{\gamma},$$

we obtain

$$\frac{1-|z|^{2\gamma}}{\gamma}\left|\frac{zh''(z)}{h'(z)}\right| \leq 1-|z|^{2\gamma} < 1 \quad (z \in U).$$

Since $\Re\beta \geq \gamma > 0$, it follows from Theorem 1 that

$$\left[\beta\int_0^z \xi^{\beta-1}h'(\xi)d\xi\right]^{\frac{1}{\beta}} \in \mathcal{S}.$$

Since

$$\begin{aligned}\left[\beta\int_0^z \xi^{\beta-1}h'(\xi)d\xi\right]^{\frac{1}{\beta}} &= \left\{\beta\int_0^z \xi^{\beta-1}\prod_{i=1}^{n}\left(\frac{f_i(\xi)}{\xi}\right)^{\frac{1}{\alpha_i}}d\xi\right\}^{\frac{1}{\beta}}\\ &= F_{\alpha_1,\alpha_2,\cdots,\alpha_n;\beta}(z),\end{aligned}$$

the function $F_{\alpha_1,\alpha_2,\cdots,\alpha_n;\beta}(z)$ defined by (3) is in $\mathcal{S}$. ∎

Theorem 6 *Let* $\alpha_1,\alpha_2,\ldots,\alpha_n,\beta \in \mathbb{C}$, $0 < \beta_i \leq 1$ $(i = 1,2,\ldots,n)$ *and*

$$\Re\beta \geq \gamma := \sum_{i=1}^{n}\frac{\beta_i}{|\alpha_i|}. \qquad (9)$$

If $f_i \in S^*_{\beta_i}$ $(i = 1,2,\cdots,n)$, *then the function* $F_{\alpha_1,\alpha_2,\cdots,\alpha_n;\beta}(z)$ *defined by* (3) *is in* $\mathcal{S}$.

Proof. From (5), we get

$$\left|\frac{zh''(z)}{h'(z)}\right| \leq \sum_{i=1}^{n} \frac{1}{|\alpha_i|} \left|\frac{zf_i'(z)}{f_i(z)} - 1\right|$$

and by using (9) and $f_i \in S^*_{\beta_i}$, we obtain

$$\left|\frac{zh''(z)}{h'(z)}\right| \leq \sum_{i=1}^{n} \frac{\beta_i}{|\alpha_i|} = \gamma.$$

This, as in the proof of Theorem 5, shows that

$$\frac{1-|z|^{2\gamma}}{\gamma} \left|\frac{zh''(z)}{h'(z)}\right| < 1 \quad (z \in U)$$

and therefore, by Theorem 1,

$$\left[\beta \int_0^z \xi^{\beta-1} h'(\xi) d\xi\right]^{\frac{1}{\beta}} \in \mathcal{S}.$$

Therefore the function $F_{\alpha_1,\alpha_2,\cdots,\alpha_n;\beta}(z)$ defined by (3) is in $\mathcal{S}$. ∎

Example 1 *Let $\alpha, \beta \in \mathbb{C}$, $0 < \beta_i \leq 1$ $(i = 1, 2, \ldots, n)$ satisfy*

$$\Re\beta \geq \frac{1}{|\alpha|} \sum_{i=1}^{n} \beta_i.$$

If $f_i \in S^*_{\beta_i}$ $(i = 1, 2, \cdots, n)$, then the function $F_{\alpha,\beta}(z)$ defined by (2) is in $\mathcal{S}$. In particular, if $\alpha, \beta \in \mathbb{C}$, $0 < \beta_1 \leq 1$ satisfy $\Re\beta \geq \frac{\beta_1}{|\alpha|}$ and $f \in S^*_{\beta_1}$, then the function $H_{\alpha,\beta}(z)$ defined by (1) is in $\mathcal{S}$.

For $i = 1, 2, \ldots, n$, let $\alpha_i, \beta \in \mathbb{C}$ and $f_i(z) \in \mathcal{A}$. Define the integral operator $T_{\alpha_1,\ldots,\alpha_n;\beta}(z)$ by

$$T_{\alpha_1,\ldots,\alpha_n;\beta}(z) = \left\{\beta \int_0^z \xi^{\beta-1} \exp\left(\sum_{i=1}^{n} \alpha_i f_i(\xi)\right) d\xi\right\}^{\frac{1}{\beta}}. \tag{10}$$

Theorem 7 *Let* $f_i \in \mathcal{A}$ $(i = 1, 2, \cdots, n)$ *satisfy*

$$|zf_i'(z)| \leq M_i \quad (M_i \geq 1; \quad z \in U).$$

Let $\alpha_i, \beta \in \mathbb{C}$ $(i = 1, 2, \ldots, n)$ *and* $\gamma > 0$ *be the smallest number such that*

$$2\sum_{i=1}^{n} M_i|\alpha_i| \leq (2\gamma+1)^{\frac{2\gamma+1}{2\gamma}}. \tag{11}$$

Then, for $\beta \in \mathbb{C}$, $\Re\beta \geq \gamma$, *the function* $T_{\alpha_1,\ldots,\alpha_n;\beta}(z)$ *defined* (10) *is in* $\mathcal{S}$.

Proof. Define the function $h(z)$ by

$$h(z) := \int_0^z \exp\left(\sum_{i=1}^{n} \alpha_i f_i(\xi)\right) d\xi.$$

Then we have $h(0) = h'(0) - 1 = 0$. Also a simple computation yields

$$h'(z) = \exp\left(\sum_{i=1}^{n} \alpha_i f_i(z)\right)$$

and

$$\frac{zh''(z)}{h'(z)} = \sum_{i=1}^{n} \alpha_i z f_i'(z). \tag{12}$$

By Schwartz's Lemma, we have $|zf_i'(z)| \leq |z|, \quad (z \in U, \quad i = 1, 2, \cdots, n)$, and therefore we obtain, from (12),

$$\left|\frac{zh''(z)}{h'(z)}\right| \leq \sum_{i=1}^{n} |\alpha_i| \ |zf_i'(z)| \leq (\sum_{i=1}^{n} M_i|\alpha_i|)|z|.$$

Thus

$$\frac{1-|z|^{2\gamma}}{\gamma}\left|\frac{zh''(z)}{h'(z)}\right| \leq \frac{|z|(1-|z|^{2\gamma})}{\gamma}\sum_{i=1}^{n} M_i|\alpha_i| \quad (z \in U).$$

For the function $Q : [0, 1] \to \mathbb{R}$ defined by $Q(t) = t(1 - t^{2\gamma})$, $\gamma > 0$, the maximum is attained at the point $t = 1/(2\gamma + 1)^{1/2\gamma}$ and thus we have

$$Q(t) \leq \frac{2\gamma}{(2\gamma+1)^{\frac{2\gamma+1}{2\gamma}}}.$$

In view of this inequality and our assumption (11), we obtain

$$\frac{1-|z|^{2\gamma}}{\gamma}\left|\frac{zh''(z)}{h'(z)}\right| \leq 1 \quad (z \in U)$$

and, by Theorem 1, $T_{\alpha_1,\ldots,\alpha_n;\beta}(z)$ is univalent in U for $\beta \in \mathbb{C}$ with $\Re\beta \geq \gamma$. ∎

Also we have the following result.

Theorem 8 *Let $\alpha_i, \beta \in \mathbb{C}$, $0 \leq \beta_i \leq 1$ $(i = 1, 2, \ldots, n)$. Let $f_i \in S^*_{\beta_i}$ $(i = 1, 2, \cdots, n)$ satisfy*

$$|f_i(z)| \leq M_i \quad (M_i \geq 1; \quad z \in U; \quad i = 1, 2, \ldots, n).$$

and $\gamma > 0$ be the smallest number such that

$$2\sum_{i=1}^{n} M_i|\alpha_i|(1+\beta_i) \leq (2\gamma+1)^{\frac{2\gamma+1}{2\gamma}}. \tag{13}$$

Then, for $\beta \in \mathbb{C}$, $\Re\beta \geq \gamma$, the function $T_{\alpha_1,\ldots,\alpha_n;\beta}(z)$ defined (10) is in $\mathcal{S}$.

Proof. From (12), we have

$$\begin{aligned}\left|\frac{zh''(z)}{h'(z)}\right| &\leq \sum_{i=1}^{n}|\alpha_i|\left|\frac{zf_i'(z)}{f_i(z)}\right||f_i(z)| \\ &\leq \sum_{i=1}^{n}|\alpha_i|\left(1+\left|\frac{zf_i'(z)}{f_i(z)}-1\right|\right)|f_i(z)| \\ &\leq \sum_{i=1}^{n} M_i|\alpha_i|(1+\beta_i)|z|.\end{aligned}$$

The remaining part of the proof is similar to the proof of Theorem 7. ∎

Similarly we have the following result.

Theorem 9 *Let $f_i \in S_{\beta_i}$ $(i = 1, 2, \cdots, n)$ satisfy*

$$|f_i(z)| \leq M_i \quad (M_i \geq 1; \quad z \in U; \quad i = 1, 2, \ldots, n).$$

Let $\alpha_i, \beta \in \mathbb{C}$, $0 \leq \beta_i \leq 1$ $(i = 1, 2, \ldots, n)$, and $\gamma > 0$ be the smallest number such that

$$2\sum_{i=1}^{n} M_i^2|\alpha_i|(1+\beta_i) \leq (2\gamma+1)^{\frac{2\gamma+1}{2\gamma}}. \tag{14}$$

Then, for $\beta \in \mathbb{C}$, $\Re\beta \geq \gamma$, the function $T_{\alpha_1,\ldots,\alpha_n;\beta}(z)$ defined (10) is in $\mathcal{S}$.

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