ON ACCELERATION AND ACCELERATION AXES IN DUAL LORENTZIAN SPACE ID_1^3

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Abstract. In this paper we have explained one parameter Dual Lorentzian Spherical Motions in three dimensional dual Lorentzian Space and given relations which are concerned with velocities and accelerations of this motion. In the original part of this paper relations have been obtained related to acceleration and acceleration axes of one parameter Dual Lorentzian Motion.

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1. Introduction

Dual numbers were introduced in the 19th century by Clifford [1] and quickly found application in description of movements of rigid bodies in three dimensions [2], [3] and in description of geometrical objects also in three dimensional space [4]. The relevant formalism was developed. It has contemporary application within the curve design methods in computer aided geometric design and computer modeling of rigid bodies, linkages, robots, mechanism design, modeling human body dynamics etc. [5], [8]. For several decades there were attempts to apply dual numbers to rigid body dynamics. Investigators showed that the momentum of a rigid body can be described as a motor that obeys the motor transformation rule; hence, its derivative with respect to time yields the dual force. However, in those investigations, while going from the velocity motor to the momentum motor, there was always a need to expand the equation to six dimensions and to treat the velocity motor as two separate real vector.

E. Study devoted special attention to the representation of directed lines by dual unit vectors and defined the mapping that is known by his name. There exists one-to-one correspondence between the vectors of dual unit sphere S^2 and the directed lines of space of lines IR^3 , [3].

Considering one and two parameters spherical motions in Euclidean space, Muller [9] has given the relations for absolute, sliding, relative velocities and pole curves of these motions. Also, one parameter dual motions in three dimensional dual space ID^3, the relations about the velocities and accelerations of these motions have been investigated. Moreover relations have been given related to acceleration and acceleration axes of one parameter dual motions, [10]. If we take Minkowski 3-space IR_1^3 instead of E^3 the E. Study mapping can be stated as follows: The dual time-like and space-like unit vectors of dual hyperbolic and Lorentzian unit spheres $\tilde{H}_0^2$ and $\tilde{S}_1^2$ at the dual Lorentzian space ID_1^3 are in one-to-one correspondence with the directed time-like and space-like lines of the space of Lorentzian lines IR_1^3 , respectively, [11].

One parameter dual Lorentzian motions in three dimensional Minkowski space IR_1^3 and the relations concerning the velocities, accelerations and acceleration axes of these motions have been given by [12]. Moreover, theorems and corollaries about velocities (absolute, sliding, relative velocities) and accelerations (absolute, sliding, relative, Coriolis accelerations) of one parameter dual Lorentzian motions in three dimensional dual Lorentzian space ID_1^3 have been obtained in [12].

We hope that these results will contribute to the study of space kinematics and physics applications.

2.Basic Concepts

If a and a^* are real numbers and $\varepsilon^2 = 0$, the combination $\tilde{a} = a + \varepsilon a^*$ is called a dual number, where ε is dual unit. The set of all dual numbers forms a commutative ring over the real number field and denoted by ID. Then the set

$$ID^3 = \left\{ \vec{\tilde{a}} = (A_1, A_2, A_3) |\, A_i \in ID, 1 \leq i \leq 3 \right\}$$

is a module over the ring which is called a ID-Module or dual space and denoted by ID^3. The elements of ID^3 are called dual vectors. Thus, a dual vector $\overrightarrow{\tilde{a}}$ can be written

$$\overrightarrow{\tilde{a}} = \overrightarrow{a} + \varepsilon \overrightarrow{a^*}$$

where $\overrightarrow{a}$ and $\overrightarrow{a^*}$ are real vectors at IR^3.

The Lorentzian inner product of dual vectors $\overrightarrow{\tilde{a}}$ and $\overrightarrow{\tilde{b}}$ in ID^3 is defined by

$$\left\langle \overrightarrow{\tilde{a}}\,, \overrightarrow{\tilde{b}} \right\rangle = \left\langle \vec{a}, \vec{b} \right\rangle + \varepsilon \left(\left\langle \vec{a}, \overrightarrow{b^*} \right\rangle + \left\langle \overrightarrow{a^*}, \vec{b} \right\rangle \right)$$

with the lorentzian inner product $\vec{a}$ and $\vec{b}$

$$\left\langle \vec{a}, \vec{b} \right\rangle = -a_1 b_1 + a_2 b_2 + a_3 b_3$$

where $\overrightarrow{a} = (a_1, a_2, a_3)$, $\overrightarrow{b} = (b_1, b_2, b_3)$. Therefore, ID^3 with the Lorentzian inner product $\left\langle \overrightarrow{\tilde{a}}\,, \overrightarrow{\tilde{b}} \right\rangle$ is called three dimensional dual Lorentzian space and denoted by of ID_1^3 , [11]. A dual vector $\overrightarrow{\tilde{a}}$ is said to be time-like if $\overrightarrow{a}$ is time-like ($\langle \overrightarrow{a}, \overrightarrow{a} \rangle < 0$), space-like if $\overrightarrow{a}$ is space-like ($\langle \overrightarrow{a}, \overrightarrow{a} \rangle > 0$ or $\overrightarrow{a} = 0$) and light-like (or null) if $\overrightarrow{a}$ is light-like (or null) ($\langle \overrightarrow{a}, \overrightarrow{a} \rangle = 0, \overrightarrow{a} \neq 0$). The set of all dual vectors such that $\left\langle \overrightarrow{\tilde{a}}\,, \overrightarrow{\tilde{a}} \right\rangle = 0$ is called the dual light-like (or null) cone and denoted by Γ. The norm of a dual vector is defined to be

$$\left\| \overrightarrow{\tilde{a}} \right\| = \| \overrightarrow{a} \| + \varepsilon \frac{\left\langle \overrightarrow{a}, \overrightarrow{a^*} \right\rangle}{\| \overrightarrow{a} \|} \qquad , \qquad \overrightarrow{a} \neq 0.$$

Let $\overrightarrow{\tilde{a}}$ and $\overrightarrow{\tilde{b}}$ be two future-pointing (respectively, past-pointing) unit dual time-like vectors in ID_1^3. Then we have

$$\left\langle \overrightarrow{\tilde{a}}\,, \overrightarrow{\tilde{b}} \right\rangle = -\cosh \Theta.$$

If $\overrightarrow{\tilde{a}}$ and $\overrightarrow{\tilde{b}}$ are two space-like unit vector and $Sp\left\{ \overrightarrow{\tilde{a}}\,, \overrightarrow{\tilde{b}} \right\}$ is time-like (i.e., the space spanned by $\overrightarrow{\tilde{a}}$ and $\overrightarrow{\tilde{b}}$ is time-like), and $Sp\left\{ \overrightarrow{\tilde{a}}\,, \overrightarrow{\tilde{b}} \right\}$ is space-like (i.e., the space spanned by $\overrightarrow{\tilde{a}}$ and $\overrightarrow{\tilde{b}}$ is space-like) then

$$\left\langle \overrightarrow{\tilde{a}}\,, \overrightarrow{\tilde{b}} \right\rangle = \cosh \Theta$$

and

$$\left\langle \overrightarrow{\tilde{a}}, \overrightarrow{\tilde{b}} \right\rangle = \cos\Theta$$

respectively, where $\Theta = \theta + \varepsilon\theta^*$ is dual angle between $\overrightarrow{\tilde{a}}$ and $\overrightarrow{\tilde{b}}$ unit dual vectors [11]. The angle Θ is formed with angle θ between directed lines $\vec{a}$, $\vec{b}$ and θ^* is the shortest Lorentzian distance between these lines.
The dual Lorentzian and dual hyperbolic unit spheres in ID_1^3 are given by

$$\tilde{S}_1^2 = \left\{ \overrightarrow{\tilde{a}} = \overrightarrow{a} + \varepsilon\overrightarrow{a^*} \in ID_1^3 \middle| \left\langle \overrightarrow{\tilde{a}}, \overrightarrow{\tilde{a}} \right\rangle = 1, \overrightarrow{a}, \overrightarrow{a^*} \in IR_1^3 \right\}$$

and

$$\tilde{H}_0^2 = \left\{ \overrightarrow{\tilde{a}} = \overrightarrow{a} + \varepsilon\overrightarrow{a^*} \in ID_1^3 \middle| \left\langle \overrightarrow{\tilde{a}}, \overrightarrow{\tilde{a}} \right\rangle = -1, \overrightarrow{a}, \overrightarrow{a^*} \in IR_1^3 \right\}$$

respectively. These are two components of $\tilde{H}_0^2$. We call the components of $\tilde{H}_0^2$ passing through $(1,0,0)$ and $(-1,0,0)$ a future-pointing dual hyperbolic unit sphere, and denote them by $\tilde{H}_0^{2+}$ and $\tilde{H}_0^{2-}$ respectively. With respect to this definition, we can write

$$\tilde{H}_0^{2+} = \left\{ \overrightarrow{\tilde{a}} = \overrightarrow{a} + \varepsilon\overrightarrow{a^*} \in \tilde{H}_0^2 \middle| \overrightarrow{a} \,\text{is a future} - \text{pointing time} - \text{like unit vector} \right\}$$

and

$$\tilde{H}_0^{2-} = \left\{ \overrightarrow{\tilde{a}} = \overrightarrow{a} + \varepsilon\overrightarrow{a^*} \in \tilde{H}_0^2 \middle| \overrightarrow{a} \text{ is a past} - \text{pointing time} - \text{like unit vector} \right\}$$

The dual Lorentzian cross-product of $\overrightarrow{\tilde{a}}$ and $\overrightarrow{\tilde{b}}$ in ID_1^3 is defined as

$$\overrightarrow{\tilde{a}} \wedge \overrightarrow{\tilde{b}} = \overrightarrow{a} \wedge \overrightarrow{b} + \varepsilon\left(\overrightarrow{a} \wedge \overrightarrow{b^*} + \overrightarrow{a^*} \wedge \overrightarrow{b} \right)$$

with the Lorentzian cross-product of $\vec{a}$ and $\vec{b}$

$$\overrightarrow{a} \wedge \overrightarrow{b} = (a_3b_2 - a_2b_3, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$$

where $\overrightarrow{a} = (a_1, a_2, a_3)$, $\overrightarrow{b} = (b_1, b_2, b_3)$, [11].

Let f be a differentiable dual function. Thus, Taylor expansion of the dual function f is given by

$$f(x+\varepsilon x^*) = f(x) + \varepsilon x^* f'(x)$$

where $f'(x)$ is the first derivatives of f, [10].

3.Dual Lorentzian Spatial Motions

The two coordinate systems which represent the moving space IL and the fixed space IL' in IR_1^3, respectively. Let us express the displacements (IL/IL') of IL with respect to IL' in a third orthonormal right-handed system (or relative system) $\left\{Q; \overrightarrow{\tilde{r}_1}, \overrightarrow{\tilde{r}_2}, \overrightarrow{\tilde{r}_3}\right\}$.

Considering E. Study theorem, it is obvious that the dual points of the unit dual Lorentz spheres $\tilde{K}'$, $\tilde{K}$ and $\tilde{K}_1$ with common centre $\tilde{M}$ correspond one to one $\overrightarrow{e_i}$, $\overrightarrow{e_i'}$ and $\overrightarrow{r_i}$ $(1 \leq i \leq 3)$ axes in dual Lorentzian space ID_1^3, respectively. Therefore, IL_1/IL, IL_1/IL' and hence IL/IL' Lorentzian motions can be considered as dual Lorentzian spherical motions $\tilde{K}_1\big/\tilde{K}$, $\tilde{K}_1\big/\tilde{K}'$ and $\tilde{K}\big/\tilde{K}'$, respectively.

Let $\tilde{K}$, $\tilde{K}'$ and $\tilde{K}_1$ be unit Lorentzian dual spheres with common centre $\tilde{M}$ and $\left\{\tilde{M}; \overrightarrow{\tilde{e}_1}, \overrightarrow{\tilde{e}_2}, \overrightarrow{\tilde{e}_3}\right\}$, $\left\{\tilde{M}; \overrightarrow{\tilde{e}_1'}, \overrightarrow{\tilde{e}_2'}, \overrightarrow{\tilde{e}_3'}\right\}$ and $\left\{\tilde{M}; \overrightarrow{\tilde{r}_1}, \overrightarrow{\tilde{r}_2}, \overrightarrow{\tilde{r}_3}\right\}$ be the orthonormal coordinate systems, respectively, which are rigidly linked to these spheres. Here

$$\overrightarrow{\tilde{e}_i} = \overrightarrow{e_i} + \varepsilon \overrightarrow{e_i^*} \quad , \quad \overrightarrow{\tilde{e}_i'} = \overrightarrow{e_i'} + \varepsilon \overrightarrow{e_i'^*}, \quad \overrightarrow{\tilde{r}_i} = \overrightarrow{r_i} + \varepsilon \overrightarrow{r_i^*} \quad , \quad 1 \leq i \leq 3$$

and

$$\overrightarrow{e_i^*} = \overrightarrow{\tilde{M}O} \wedge \overrightarrow{e_i} \quad , \quad \overrightarrow{e_i'^*} = \overrightarrow{\tilde{M}O'} \wedge \overrightarrow{e_i'} \quad , \quad \overrightarrow{r_i^*} = \overrightarrow{\tilde{M}Q} \wedge \overrightarrow{r_i} \quad , \quad 1 \leq i \leq 3.$$

Since each of these systems are oriented to the same direction, one can write

$$\tilde{R} = \tilde{A}\tilde{E}, \qquad \tilde{R} = \tilde{A}'\tilde{E}' \tag{3.1}$$

where $\tilde{A}$ and $\tilde{A}'$ represent a special dual orthogonal matrices. Here

$$\tilde{R} = \begin{bmatrix} \overrightarrow{\tilde{r}_1} \\ \overrightarrow{\tilde{r}}_2 \\ \overrightarrow{\tilde{r}}_3 \end{bmatrix}, \qquad \tilde{E} = \begin{bmatrix} \overrightarrow{\tilde{e}_1} \\ \overrightarrow{\tilde{e}_2} \\ \overrightarrow{\tilde{e}_3} \end{bmatrix}, \qquad \tilde{E}' = \begin{bmatrix} \overrightarrow{\tilde{e}_1'} \\ \overrightarrow{\tilde{e}_2'} \\ \overrightarrow{\tilde{e}_3'} \end{bmatrix}.$$

The elements of positive dual orthogonal matrices $\tilde{A}$ and $\tilde{A}'$ are differentiable functions of dual parameter $\tilde{t} = t+\varepsilon t^*$. Throughout this paper we will take $t^* = 0$, unless otherwise mentioned. Hence one-parameter motion is determined by the matrix $\tilde{A}$ (or$\tilde{A}'$) and called as one-parameter dual Lorentzian spherical motion.
If we consider equation (3.1), then differential of the relative orthonormal coordinate system $\tilde{R}$ with respect to $\tilde{K}$ and $\tilde{K}'$ are

$$d\tilde{R} = \tilde{\Omega}\tilde{R}, \qquad d'\tilde{R} = \tilde{\Omega}'\tilde{R} \tag{3.2}$$

where $\tilde{\Omega} = d\tilde{A}\tilde{A}^{-1}$ and $\tilde{\Omega}' = d\tilde{A}'\left(\tilde{A}'\right)^{-1}$, respectively, [12]. The matrices $\tilde{\Omega}$ and $\tilde{\Omega}'$ are anti-symmetric matrices in the sense of Lorentzian. Let us denote the permutations of the indices $i, j, k = 1, 2, 3; 2, 3, 1; 3, 1, 2;$ by $\tilde{\Omega}_{ij} = \tilde{\Omega}_k$. Then we can get that

$$\tilde{\Omega} = \begin{bmatrix} 0 & \tilde{\Omega}_3 & -\tilde{\Omega}_2 \\ \tilde{\Omega}_3 & 0 & -\tilde{\Omega}_1 \\ -\tilde{\Omega}_2 & \tilde{\Omega}_1 & 0 \end{bmatrix}, \qquad \tilde{\Omega}_i = \Omega_i + \varepsilon\Omega_i^*, \qquad 1 \le i \le 3.$$

In the similar way, the matrix $\tilde{\Omega}'$ is obtained to be

$$\tilde{\Omega}' = \begin{bmatrix} 0 & \tilde{\Omega}_3' & -\tilde{\Omega}_2' \\ \tilde{\Omega}_3' & 0 & -\tilde{\Omega}_1' \\ -\tilde{\Omega}_2' & \tilde{\Omega}_1' & 0 \end{bmatrix}, \qquad \tilde{\Omega}_i' = \Omega_i' + \varepsilon\Omega_i'^{\,*}, \qquad 1 \le i \le 3.$$

where $\tilde{\Omega}_i = \Omega_i + \varepsilon\Omega_i^*$ and $\tilde{\Omega}_i' = \Omega_i' + \varepsilon\Omega_i'^{\,*}$ $(1 \le i \le 3)$ are dual Pfaffian forms. Let us consider a point $\tilde{X}$ on unit dual Lorentz sphere $\tilde{K}_1$. The coordinates of the point $\tilde{X}$ is $\tilde{X}_i = x_i + \varepsilon x_i^*$ $(1 \le i \le 3)$. Thus differential of the vector $\overrightarrow{O\tilde{X}} = \overrightarrow{\tilde{X}} = \tilde{X}^T\tilde{R}$ with respect to moving unit dual Lorentz sphere $\tilde{K}$ is

$$d\overrightarrow{\tilde{X}} = \left(d\tilde{X}^T + \tilde{X}^T\tilde{\Omega}\right)\tilde{R}. \tag{3.3}$$

Therefore, the relative velocity vector of $\tilde{X}$ becomes $\overrightarrow{\tilde{V}_r} = {d\overrightarrow{\tilde{x}}}\big/{dt}$. Similarly the differential of point $\tilde{X}$ with respect to fixed unit dual Lorentz sphere $\tilde{K}'$ is

$$d'\overrightarrow{\tilde{X}} = \left(d\tilde{X}^T + \tilde{X}^T\tilde{\Omega}'\right)\tilde{R}. \tag{3.4}$$

So, the absolute velocity vector is expressed to be $\overrightarrow{\tilde{V}_a} = d'\overrightarrow{\tilde{X}} \Big/ dt$. If the point $\tilde{X}$ is fixed on moving unit dual Lorentz sphere $\tilde{K}$, since $\overrightarrow{\tilde{V}_r} = 0$, we reach

$$d\tilde{X}^T = -\tilde{X}^T\tilde{\Omega}. \tag{3.5}$$

Substituting equation (3.5) into equation (3.4) we find for the sliding velocity of $\tilde{X}$ that

$$\overrightarrow{\tilde{V}_f} = d_f\overrightarrow{\tilde{X}} = \tilde{X}^T\left(\tilde{\Omega}' - \tilde{\Omega}\right)\tilde{R}. \tag{3.6}$$

If, at this point, we consider the dual vector

$$\overrightarrow{\tilde{\Psi}} = -\tilde{\Psi}_1\overrightarrow{\tilde{r}_1} + \tilde{\Psi}_2\overrightarrow{\tilde{r}_2} + \tilde{\Psi}_3\overrightarrow{\tilde{r}_3}$$

in which the components are $\tilde{\Psi}_i = \tilde{\Omega}'_i - \tilde{\Omega}_i$ $(1 \leq i \leq 3)$, then equation (3.6) reduces to

$$d_f\overrightarrow{\tilde{X}} = \overrightarrow{\tilde{V}_f} = \overrightarrow{\tilde{\Psi}} \wedge \overrightarrow{\tilde{X}}, \tag{3.7}$$

where $\overrightarrow{\tilde{\Psi}} = \overrightarrow{\psi} + \varepsilon\overrightarrow{\psi^*}$ is a dual Lorentzian Pfaffian vector. The real part $\overrightarrow{\psi}$ and the dual part $\overrightarrow{\psi^*}$ of $\overrightarrow{\tilde{\Psi}}$ correspond to the rotation motions and the translation motions of the spatial motion.

4. Acceleration and Acceleration Axes

From equation (3.7) it is easily seen that the sliding acceleration of dual vector $\overrightarrow{\tilde{X}}$ is obtained as follows.

$$\overrightarrow{\tilde{J}} = d\left(d_f\overrightarrow{\tilde{X}}\right) = \dot{\overrightarrow{\tilde{\Psi}}} \wedge \overrightarrow{\tilde{X}} + \left\langle \overrightarrow{\tilde{\Psi}}, \overrightarrow{\tilde{\Psi}} \right\rangle \overrightarrow{\tilde{X}} - \left\langle \overrightarrow{\tilde{\Psi}}, \overrightarrow{\tilde{X}} \right\rangle \overrightarrow{\tilde{\Psi}} \tag{4.1}$$

where $\dot{\overrightarrow{\tilde{\Psi}}} = d\overrightarrow{\tilde{\Psi}}$ is the instantaneous dual angular acceleration vector. Taking equation (3.6) and (4.1) into consideration, we find sliding velocity and sliding acceleration in matrix form as follows

$$d_f\tilde{X} = \tilde{M}\tilde{X} \tag{4.2}$$

and

$$\tilde{J} = \left(\dot{\tilde{M}} + \tilde{M}^2\right)\tilde{X} \tag{4.3}$$

respectively. Here $\tilde{M} = \left(\tilde{\Omega}' - \tilde{\Omega}\right)^T$. From equation (4.3) it is clear that the components of dual Lorentzian acceleration $\tilde{J}$ are homogenous linear functions such that the coordinates of $\overrightarrow{\tilde{X}}$ are $\tilde{X}_i$ $(1 \leq i \leq 3)$. Hence the determinant $\tilde{D}$ of coefficients matrix of equation (4.3) is

$$\tilde{D} = \det\left(\dot{\tilde{M}} + \tilde{M}^2\right) = \left\|\overrightarrow{\tilde{\Psi}} \wedge \overrightarrow{\dot{\tilde{\Psi}}}\right\|^2 = \left\|\overrightarrow{\tilde{\Psi}}\right\|^2 \left\|\overrightarrow{\dot{\tilde{\Psi}}}\right\|^2 \sinh^2\tilde{\nabla} = \tilde{\Psi}^2\,\dot{\tilde{\Psi}}^{\,2}\sinh^2\tilde{\nabla} \tag{4.4}$$

where

$$\tilde{\nabla} = \tilde{\alpha} + \varepsilon\tilde{\alpha}^* \tag{4.5}$$

is a dual Lorentz angle between the dual space-like vectors such that the space spanned by these two dual space-like vectors is time-like. $\left\|\overrightarrow{\tilde{\Psi}}\right\| = \psi + \varepsilon\psi^* = \tilde{\Psi}$ is called instantaneous dual Lorentzian rotation angle, where $\psi = \left\|\vec{\psi}\right\|$ and $\psi^* = \frac{\langle\vec{\psi},\vec{\psi}^*\rangle}{\|\vec{\psi}\|}$. If both vectors $\overrightarrow{\tilde{\Psi}}$ and $\overrightarrow{\dot{\tilde{\Psi}}}$ correspond to the same line, then this line has no acceleration. It is very clear that in this case $\tilde{D} = 0$.

Definition 1. *If a unit dual vector $\tilde{X}$ of the unit dual Lorentz sphere and its dual acceleration vector $\overrightarrow{\tilde{J}}$ are linearly dependent, then the point $\tilde{X}$ is called dual acceleration pole and line $\overrightarrow{\tilde{X}}$ is called acceleration axis for the dual Lorentzian motion.*

If we denote the dual Lorentzian acceleration vector $\overrightarrow{\tilde{X}}$ as $\overrightarrow{\tilde{V}}$ and consider equation (4.1) then we obtain

$$\left\langle \overrightarrow{\tilde{\Psi}}, \overrightarrow{\tilde{V}} \right\rangle \overrightarrow{\tilde{\Psi}} - \overrightarrow{\dot{\tilde{\Psi}}} \wedge \overrightarrow{\tilde{V}} = \Lambda\tilde{\Psi}^2\overrightarrow{\tilde{V}} \tag{4.6}$$

where

$$\tilde{\Lambda} = \lambda + \varepsilon\lambda^* \tag{4.7}$$

is a dual scalar. Equation (4.6) correspond to three homogeneous linear equations depend on the coordinates $\tilde{X}_1$, $\tilde{X}_2$, $\tilde{X}_3$ of $\overrightarrow{\tilde{V}}$. Thus, the determinant of the coefficients matrix must be zero for non-zero solutions. So

$$\tilde{\Lambda}^3 - \tilde{\Lambda}^2 - \tilde{K}\tilde{\Lambda} + \tilde{K}\cosh^2\tilde{\nabla} = 0 \tag{4.8}$$

where

$$\tilde{K} = k + \varepsilon k^* = \frac{\dot{\tilde{\Psi}}^{\,2}}{\tilde{\Psi}^4} = \frac{\dot{\psi}^2}{\psi^4} - \varepsilon\frac{2\,\dot{\psi}\left(2\psi^*\,\dot{\psi} - \dot{\psi}^*\,\psi\right)}{\psi^5} \tag{4.9}$$

Considering

$$\begin{array}{lll} \tilde{\Lambda} = \lambda + \varepsilon\lambda^*\,, \quad \tilde{\Lambda}^2 = \lambda^2 + 2\varepsilon\lambda\lambda^* & , & \tilde{\Lambda}^3 = \lambda^3 + 3\varepsilon\lambda^2\lambda^* \\ \cosh\tilde{\nabla} = \cosh\alpha + \varepsilon\alpha^*\sinh\alpha & , & \cosh^2\tilde{\nabla} = \cosh^2\alpha + \varepsilon\alpha^*\sinh 2\alpha \end{array} \tag{4.10}$$

then equation (4.8) reduces to the following equations:

$$\begin{array}{lr} \lambda^3 - \lambda^2 - k\lambda + k\cosh^2\alpha = 0 & \text{(Real part)} \\ \lambda^* = \frac{-k^*\cosh^2\alpha + k^*\lambda - k\alpha^*\sinh 2\alpha}{3\lambda^2 - 2\lambda - k} & \text{(Dual part).} \end{array} \tag{4.11}$$

Since equation (4.8) or equation (4.11) have generally three roots as $\tilde{\Lambda}_1$, $\tilde{\Lambda}_2$, $\tilde{\Lambda}_3$ so there are three lines ℓ_1, ℓ_2 and ℓ_3 which are called instantaneous dual Lorentzian acceleration axes.

These three axes are skew lines in space ID_1^3.

The special case of $k^* = 0$ is of importance. From equation (4.9) we see that there are three distinct situation:

i) $\dot{\psi} = 0$,

ii) $2\psi^*\,\dot{\psi} - \dot{\psi}^*\,\psi = 0$,

iii) $\psi^* = 0$.

Now let us investigate these special cases individually.

i) If $\dot{\psi} = d\psi = 0$ then $\psi =$ constant and $k = 0$. Therefore, the roots of equation (4.8) find to be $\tilde{\Lambda}_1 = \tilde{\Lambda}_2 = 0$ and $\tilde{\Lambda}_3 = 1$. So, considering equation (4.8), we find the acceleration axes of one-parameter dual Lorentzian motion as

$$\ell_1 = \ell_2 = \left\langle \overrightarrow{\tilde{\Psi}}, \overrightarrow{\tilde{V}_{1,2}} \right\rangle \overrightarrow{\tilde{\Psi}} - \dot{\overrightarrow{\tilde{\Psi}}} \wedge \overrightarrow{\tilde{V}_{1,2}} = 0$$

and

$$\ell_3 = \left\langle \overrightarrow{\tilde{\Psi}}, \overrightarrow{\tilde{V}_3} \right\rangle \overrightarrow{\tilde{\Psi}} - \dot{\overrightarrow{\tilde{\Psi}}} \wedge \overrightarrow{\tilde{V}_3} - \tilde{\Psi}^2 \overrightarrow{\tilde{V}_3} = 0$$

ii) If $2\psi^* \dot{\psi} - \dot{\psi}^* \psi = 0$ then $\psi^* = c_1 \psi^2$. Therefore, the pitch of the instantaneous helicoidal motion is $\frac{\psi^*}{\psi} = c_1 \psi$, where c_1 is a constant. So the orbit of a point during instantaneous motion is a circular helix.

iii) If $\psi^* = 0$ then $d\psi^* = \dot{\psi}^* = 0$, i.e. $k^* = 0$. In this case, there is a fixed point on instantaneous dual axis $\overrightarrow{\tilde{\Psi}}$. As ψ^* corresponds to the translational part of the dual Lorentzian motion, this special case is a Lorentzian spherical motion. Thus, acceleration axes form a pencil of lines whose vertex is the centre of the sphere.

Now we consider equation (4.11) . From equation (4.11) we see that the three λ_i (and also three λ_i^*) are either all real or two of them are imaginary. To discuss the roots we define a new unknown $\tilde{S}$ as follows:

$$\tilde{\Lambda} = \tilde{S} + \tfrac{1}{3}. \qquad (4.12)$$

Therefore, substituting the last equation into equation (4.8), we find

$$\tilde{S}^3 - \tilde{B}\tilde{S} - \tilde{C} = 0 \qquad (4.13)$$

where

$$\begin{array}{lll} \tilde{S} = \mu + \varepsilon\mu^*, & \tilde{B} = b + \varepsilon b^* = \tilde{K} + \frac{1}{3}, & \tilde{C} = c + \varepsilon c^* = \left(\frac{1}{3} - \cosh^2 \tilde{\nabla}\right) \tilde{K} + \frac{2}{27}, \\ b = k + \frac{1}{3}, & b = k + \frac{1}{3}, & b^* = k^* \\ c = \frac{1}{3}k - \cosh^2 \alpha k + \frac{2}{27}, & & c^* = \left(\frac{1}{3} - \cosh^2 \alpha\right) k^* - k\alpha^* \sinh 2\alpha. \end{array} \qquad (4.14)$$

If we separate the real and imaginary parts of equation (4.13), we reach

$$\mu^3 - b\mu - c = 0$$

and

$$\mu^* = \tfrac{b^*\mu + c^*}{3\mu^2 - b} \qquad (4.15)$$

respectively. The roots of the cubic equation of μ are real and the values of μ^* are also real iff

$$4b^3 + 27c^2 \leq 0$$

or, from equation (4.14)

$$k\left(4k^2+7k+\frac{8}{3}-18k\cosh^2\alpha-4\cosh^2\alpha+27\cosh^4\alpha\right)+\frac{8}{27}\leq 0.$$

Therefore, the following theorem can be given.

Theorem 1. *In three dimensional dual Lorentzian space ID_1^3, three acceleration axes of one-parameter dual Lorentzian motion is real iff*

$$k\left(4k^2+7k+\frac{8}{3}-18k\cosh^2\alpha-4\cosh^2\alpha+27\cosh^4\alpha\right)+\frac{8}{27}\leq 0.$$

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