SUFFICIENT CONDITIONS FOR UNIVALENCE OF A GENERAL INTEGRAL OPERATOR

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Abstract. Using Dziok-Srivastava operator, we define a new family of integral operators. For this general integral operator we study some interesting univalence properties, to which a number of univalent conditions would follow upon specializing the parameters involved.

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1. Introduction, Definitions And Preliminaries

Let $\mathcal{A}_1 = \mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad a_n \geq 0, \tag{1}$$

which are analytic in the open disc $\mathcal{U} = \{z : z \in \mathbb{C} \mid z |< 1\}$ and $\mathcal{S}$ be the class of function $f(z) \in \mathcal{A}$ which are univalent in $\mathcal{U}$.

For $\alpha_j \in \mathbb{C}$ $(j = 1, 2, \ldots, q)$ and $\beta_j \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$ $(j = 1, 2, \ldots, s)$, the generalized hypergeometric function ${}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$ is defined by the infinite series

$${}_qF_s(\alpha_1, \alpha_2, \ldots, \alpha_q; \beta_1, \beta_2, \ldots, \beta_s; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ \ldots \ (\alpha_q)_n}{(\beta_1)_n \ \ldots \ (\beta_s)_n} \frac{z^n}{n!}$$

$$(q \leq s + 1;\ q, s \in N_0 = \{0, 1, 2, \ldots\};\ z \in \mathcal{U}),$$

where $(x)_n$ is the Pochhammer symbol defined by

$$(x)_n = \begin{cases} 1 & if\, n = 0 \\ x(x+1)(x+2)\ \ldots\ (x+n-1) & if\, n \in N = \{1, 2, , \ldots\}. \end{cases}$$

Corresponding to a function $h(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$ defined by

$$h(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) := z\, {}_qF_s(\alpha_1, \alpha_2, \ldots, \alpha_q; \beta_1, \beta_2, \ldots, \beta_s; z),$$

the Dziok and Srivastava operator $H(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)f(z)$ is defined by the Hadamard product

$$\begin{aligned} H(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)f(z) &:= h(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) * f(z) \\ &= z + \sum_{n=2}^{\infty} \frac{(\alpha_1)_{n-1} \ldots (\alpha_q)_{n-1}}{(\beta_1)_{n-1} \ldots (\beta_s)_{n-1}} \frac{a_n\, z^n}{(n-1)!} \end{aligned} \tag{2}$$

For convenience, we write

$$H_s^q(\alpha_1, \beta_1)f := H(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)f(z).$$

The linear operator $H_s^q(\alpha_1, \beta_1)f$ includes (as its special cases) various other linear operators which were introduced and studied by Hohlov, Carlson and Shaffer, Ruscheweyh. For details see [1].

Using Dziok-Srivastava operator, we now introduce the following:

Definition 1. *For $\gamma \in \mathbb{C}$. We now define the integral operator $F_\gamma(\alpha_1, \beta_1; z) : \mathcal{A}^n \longrightarrow \mathcal{A}$*

$$F_\gamma(\alpha_1, \beta_1; z) = \left(\big(1 + n(\gamma - 1)\big) \int_0^z \big(H_s^q(\alpha_1, \beta_1)f_1(t)\big)^{\gamma-1} \ldots \big(H_s^q(\alpha_1, \beta_1)f_n(t)\big)^{\gamma-1} dt \right)^{\frac{1}{1+n(\gamma-1)}}, \tag{3}$$

where $f_i \in \mathcal{A}$ and $H_s^q(\alpha_1, \beta_1)f(z)$ is the Dziok-Srivastava operator.

Remark 1. It is interesting to note that the integral operator $F_\gamma(\alpha_1, \beta_1; z)$ generalizes many operators which were introduced and studied recently. Here we list a few of them

1. Let $q = 2$, $s = 1$, $\alpha_1 = \beta_1$ and $\alpha_2 = 1$, then the operator $F_\gamma(\alpha_1, \alpha_1; z)$ reduces to an integral operator

$$F_\gamma(z) = \left[(n(\gamma-1)+1)\int_0^z \left(f_1(t)\right)^{\gamma-1} \dots \left(f_n(t)\right)^{\gamma-1} dt.\right]^{\frac{1}{(n(\gamma-1)+1)}}, \quad (4)$$

studied by D.Breaz and N.Breaz [2].

2. When $q = 2$, $s = 1$, $\alpha_1 = 2$, $\beta_1 = 1$ and $\alpha_2 = 1$, the integral operator $F_\gamma(2, 1; z)$ reduces to an operator of the form

$$F_\gamma(z) = \left[(n(\gamma-1)+1)\int_0^z t^{n(\gamma-1)}\left(f_1'(t)\right)^{\gamma-1} \dots \left(f_n'(t)\right)^{\gamma-1} dt.\right]^{\frac{1}{(n(\gamma-1)+1)}}, \quad (5)$$

where $\gamma \geq 0$.

We now state the following lemma which we need to establish our results in the sequel.

Lemma 1.[3] *If $f \in \mathcal{A}$ satisfies the condition*

$$\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| \leq 1, \quad \text{for all } z \in \mathcal{U}, \quad (6)$$

then the function $f(z)$ is univalent in $\mathcal{U}$.

Lemma 2.(Schwartz Lemma) *Let $f \in \mathcal{A}$ satisfy the condition $|f(z)| \leq 1$, for all $z \in \mathcal{U}$, then*

$$|f(z)| \leq |z|, \quad \text{for all } z \in \mathcal{U},$$

and equality holds only if $f(z) = \epsilon z$, where $|\epsilon| = 1$.

Pascu [4, 5] has proved the following result

Lemma 3. *Let $\beta \in \mathbb{C}$, $Re\beta \geq \alpha > 0$. If $f \in \mathcal{A}$ satisfies*

$$\frac{1-|z|^{2Re\,\alpha}}{Re\,\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1 \quad (z \in \mathcal{U}),$$

then the integral operator

$$F_\beta(z) = \left[\beta \int_0^z t^{\beta-1} f'(t)\, dt\right]^{\frac{1}{\beta}}$$

is univalent.

Lemma 4. *Let $g \in \mathcal{A}$ satisfies the condition (6) and let α be a complex number with $\mid \alpha - 1 \mid \leq \frac{Re\,\alpha}{3}$. If $\mid g(z) \mid \leq 1$, $\forall\, z \in \mathcal{U}$ then the function*

$$G_\alpha(z) = \left(\alpha \int_0^z g^{(\alpha-1)}(t)dt\right)^{\frac{1}{\alpha}}$$

belongs to $\mathcal{S}$.

2.Main Results

Theorem 1. *Let $f_i \in \mathcal{A}$, $\gamma \in \mathbb{C}$. If*

$$\left|\frac{z^2(H_s^q(\alpha_1,\beta_1)f_i(z))'}{(H_s^q(\alpha_1,\beta_1)f_i(z))^2} - 1\right| \leq 1, \quad \mid \gamma - 1 \mid \leq \frac{Re\,\gamma}{3n}, \quad \text{and} \mid H_s^q(\alpha_1,\beta_1)f_i(z) \mid \leq 1$$

for all $z \in \mathcal{U}$ then $F_\gamma(\alpha_1,\beta_1;z)$ given by (3) is univalent.

Proof. From the definition of the operator $H_s^q(\alpha_1,\beta_1)f(z)$, it can be easily seen that

$$\frac{H_s^q(\alpha_1,\beta_1)f(z)}{z} \neq 0 \quad (z \in \mathcal{U})$$

and moreover for $z = 0$, we have

$$\left(\frac{H_s^q(\alpha_1,\beta_1)f_1(z)}{z}\right)^{\gamma-1} \cdots \left(\frac{H_s^q(\alpha_1,\beta_1)f_n(z)}{z}\right)^{\gamma-1} = 1$$

From (3), we have

$$F_\gamma(\alpha_1,\beta_1;z) = \left(\big(1+n(\gamma-1)\big)\int_0^z t^{n(\gamma-1)}\left(\frac{H_s^q(\alpha_1,\beta_1)f_1(t)}{t}\right)^{\gamma-1} \cdots \right.$$
$$\left.\left(\frac{H_s^q(\alpha_1,\beta_1)f_n(t)}{t}\right)^{\gamma-1} dt\right)^{\frac{1}{1+n(\gamma-1)}}.$$

We consider the function

$$h(z) = \int_0^z \left(\frac{H_s^q(\alpha_1,\beta_1)f_1(t)}{t}\right)^{\gamma-1} \cdots \left(\frac{H_s^q(\alpha_1,\beta_1)f_n(t)}{t}\right)^{\gamma-1} dt \qquad (7)$$

The function h is regular in $\mathcal{U}$ and from (7) we obtain

$$h^{'}(z) = \left(\frac{H_s^q(\alpha_1,\beta_1)f_1(z)}{z}\right)^{\gamma-1}\cdots\left(\frac{H_s^q(\alpha_1,\beta_1)f_n(z)}{z}\right)^{\gamma-1}$$

and

$$h^{''}(z) = (\gamma-1)\left(\frac{(H_s^q(\alpha_1,\beta_1)f_1(z))^{'}}{H_s^q(\alpha_1,\beta_1)f_1(z)} - \frac{1}{z}\right)h^{'}(z) + \ldots$$
$$+(\gamma-1)\left(\frac{(H_s^q(\alpha_1,\beta_1)f_n(z))^{'}}{H_s^q(\alpha_1,\beta_1)f_n(z)} - \frac{1}{z}\right)h^{'}(z)$$

From the above inequalities we have

$$\frac{zh^{''}(z)}{h^{'}(z)} = (\gamma-1)\left(\frac{z(H_s^q(\alpha_1,\beta_1)f_1(z))^{'}}{H_s^q(\alpha_1,\beta_1)f_1(z)} - 1\right) + \ldots \tag{8}$$
$$+(\gamma-1)\left(\frac{z(H_s^q(\alpha_1,\beta_1)f_n(z))^{'}}{H_s^q(\alpha_1,\beta_1)f_n(z)} - 1\right).$$

On multiplying the modulus of equation (8) by $\frac{1-|z|^{2Re\,\gamma}}{Re\,\gamma} > 0$, we obtain

$$\frac{1-\mid z\mid^{2Re\,\gamma}}{Re\,\gamma}\left|\frac{zh^{''}(z)}{h^{'}(z)}\right| \leq \frac{1-\mid z\mid^{2Re\,\gamma}}{Re\,\gamma}\Bigg[\mid\gamma-1\mid\left(\left|\frac{z(H_s^q(\alpha_1,\beta_1)f_1(z))^{'}}{H_s^q(\alpha_1,\beta_1)f_1(z)}\right|+1\right)+$$
$$\ldots+\mid\gamma-1\mid\left(\left|\frac{z(H_s^q(\alpha_1,\beta_1)f_n(z))^{'}}{H_s^q(\alpha_1,\beta_1)f_n(z)}\right|+1\right)\Bigg]$$
$$\leq\mid\gamma-1\mid\frac{1-\mid z\mid^{2Re\,\gamma}}{Re\,\gamma}\left(\left|\frac{z^2(H_s^q(\alpha_1,\beta_1)f_1(z))^{'}}{(H_s^q(\alpha_1,\beta_1)f_1(z))^2}\right|\frac{\mid H_s^q(\alpha_1,\beta_1)f_1(z)\mid}{\mid z\mid}+1\right)+\ldots$$
$$+\mid\gamma-1\mid\frac{1-\mid z\mid^{2Re\,\gamma}}{Re\,\gamma}\left(\left|\frac{z^2(H_s^q(\alpha_1,\beta_1)f_n(z))^{'}}{(H_s^q(\alpha_1,\beta_1)f_n(z))^2}\right|\frac{\mid H_s^q(\alpha_1,\beta_1)f_n(z)\mid}{\mid z\mid}+1\right).$$

Since $H_s^q(\alpha_1,\beta_1)f(z)$ satisfies the conditions of the Schwartz Lemma, on applying the same on the above inequality, we obtain

$$\frac{1-\mid z\mid^{2Re\,\gamma}}{Re\,\gamma}\left|\frac{zh''(z)}{h'(z)}\right| \leq \mid\gamma-1\mid \frac{1-\mid z\mid^{2Re\,\gamma}}{Re\,\gamma}\left(\left|\frac{z^2(H_s^q(\alpha_1,\beta_1)f_1(z))'}{(H_s^q(\alpha_1,\beta_1)f_1(z))^2}-1\right|+2\right)+$$

$$\ldots + \mid\gamma-1\mid \frac{1-\mid z\mid^{2Re\,\gamma}}{Re\,\gamma}\left(\left|\frac{z^2(H_s^q(\alpha_1,\beta_1)f_n(z))'}{(H_s^q(\alpha_1,\beta_1)f_n(z))^2}-1\right|+2\right)$$

$$\leq \frac{3\mid\gamma-1\mid}{Re\,\gamma} + \ldots + \frac{3\mid\gamma-1\mid}{Re\,\gamma} = \frac{3n\mid\gamma-1\mid}{Re\,\gamma}.$$

But $\mid\gamma-1\mid\leq\frac{Re\,\gamma}{3n}$, so we obtain for all $z\in\mathcal{U}$

$$\frac{1-\mid z\mid^{2Re\,\gamma}}{Re\,\gamma}\left|\frac{zh''(z)}{h'(z)}\right| \leq 1. \tag{9}$$

It follows from Lemma 3 that

$$\left(\big(1+n(\gamma-1)\big)\int_0^z t^{n(\gamma-1)}h'(t)\,dt\right)^{\frac{1}{1+n(\gamma-1)}} \in \mathcal{S}$$

Since

$$\left(\big(1+n(\gamma-1)\big)\int_0^z t^{n(\gamma-1)}h'(t)\,dt\right)^{\frac{1}{1+n(\gamma-1)}} =$$

$$\left(\big(1+n(\gamma-1)\big)\int_0^z \big(H_s^q(\alpha_1,\beta_1)f_1(t)\big)^{\gamma-1}\ldots\big(H_s^q(\alpha_1,\beta_1)f_n(t)\big)^{\gamma-1}\,dt\right)^{\frac{1}{1+n(\gamma-1)}}$$

$$= F_\gamma(\alpha_1,\beta_1;z),$$

hence $F_\gamma(\alpha_1,\beta_1;z)\in\mathcal{S}$.

Putting $q=2$, $s=1$, $\alpha_1=\beta_1$ and $\alpha_2=1$, in Theorem 1, we have

Corollary 1. [2] *Let* $f_i\in\mathcal{A}$, $\gamma\in\mathbb{C}$. *If*

$$\left|\frac{z^2f_i'(z)}{(f_i(z))^2}-1\right|\leq 1, \quad \mid\gamma-1\mid\leq\frac{Re\,\gamma}{3n}, \quad and \mid f_i(z)\mid\leq 1$$

for all $z \in \mathcal{U}$ then the operator

$$F_\gamma(z) = \left[(n(\gamma-1)+1)\int_0^z \big(f_1(t)\big)^{\gamma-1} \dots \big(f_n(t)\big)^{\gamma-1} dt\right]^{\frac{1}{(n(\gamma-1)+1)}}$$

is univalent.

Theorem 2. *Let $f \in \mathcal{A}$ satisfy*

$$\left|\frac{z^2(H_s^q(\alpha_1,\beta_1)f(z))'}{(H_s^q(\alpha_1,\beta_1)f(z))^2} - 1\right| < 1, \quad \forall\, z \in \mathcal{U}$$

and $\gamma \in \mathbb{C}$ with $\mid \gamma - 1 \mid \leq \frac{Re\,\gamma}{3k}$. If $\mid H_s^q(\alpha_1,\beta_1)f(z) \mid \leq 1$, $\forall\, z \in \mathcal{U}$, then the function

$$F_\gamma^k(\alpha_1,\beta_1;z) = \left(\big(k(\gamma-1)+1\big)\int_0^z (H_s^q(\alpha_1,\beta_1)f(t))^{k(\gamma-1)}\, dt\right)^{\frac{1}{k(\gamma-1)+1}} \tag{10}$$

is univalent.

Proof. From (10) we have

$$F_\gamma^k(\alpha_1,\beta_1;z) = \left(\big(k(\gamma-1)+1\big)\int_0^z t^{k(\gamma-1)}\left(\frac{H_s^q(\alpha_1,\beta_1)f(t)}{t}\right)^{k(\gamma-1)} dt\right)^{\frac{1}{k(\gamma-1)+1}}$$

and let

$$h(z) = \int_0^z \left(\frac{H_s^q(\alpha_1,\beta_1)f(t)}{t}\right)^{k(\gamma-1)} dt.$$

From here using arguments similar to those detailed in Theorem 1, we can prove the assertion of the Theorem 2.

We note that on restating the Theorem 2 for the choice of $q = 2$, $s = 1$, $\alpha_1 = \beta_1$ and $\alpha_2 = 1$, we have the result proved by D.Breaz and N.Breaz [2].

References

[1] R. Aghalary et al., *Subordinations for analytic functions defined by the Dziok-Srivastava linear operator*, Appl. Math. Comput., 187, (2007), no. 1, 13–19.

[2] D. Breaz and N. Breaz, *An integral univalent operator*, Acta Math. Univ. Comenian. (N.S.), 7, (2007), no. 2, 137–142.

[3] S. Ozaki and M. Nunokawa, *The Schwarzian derivative and univalent functions*, Proc. Amer. Math. Soc., 33, (1972), 392–394.

[4] N. N. Pascu, *On a univalence criterion. II*, in *Itinerant seminar on functional equations, approximation and convexity (Cluj-Napoca, 1985)*, 153–154, Univ. "Babeş-Bolyai", Cluj.

[5] N. N. Pascu, *An improvement of Becker's univalence criterion*, in *Proceedings of the Commemorative Session: Simion Stoïlow (Braşov, 1987)*, 43–48, Univ. Braşov, Braşov.

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