THE INTEGRAL OPERATOR ON THE SP CLASS

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Abstract. Let SP be the subclass of S consisting of all analytic and univalent functions $f(z)$ in the open unit disk U with $f(0) = 0$ and $f'(0) = 1$. For $f_j(z) \in SP$, an integral operator $F_n(z)$ is introduced. The aim of the present paper is to discuss the order of convexity for $F_n(z)$.

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1. Introduction

Let $U = \{z \in C, |z| < 1\}$ be the unit disc of the complex plane and denote by $H(U)$, the class of the holomorphic functions in U. Consider

$$A = \left\{f \in H(U), f(z) = z + a_2z^2 + a_3z^3 + ..., z \in U\right\}$$

be the class of analytic functions in U and $S = \{f \in A : f \text{ is univalent in } U\}$.

Denote with K the class of the holomorphic functions in U with $f(0) = f'(0) - 1 = 0$, where is convex functions in U , defined by

$$K = \left\{f \in H(U) : f(0) = f'(0) - 1 = 0, \mathbf{Re}\left\{\frac{zf''(z)}{f'(z)} + 1\right\} > 0, z \in U\right\}.$$

A function $f \in A$ is the convex function of order $\alpha, 0 \leq \alpha < 1$ and denote this class by $K(\alpha)$ if f verify the inequality

$$\mathbf{Re}\left\{\frac{zf''(z)}{f'(z)} + 1\right\} > \alpha, z \in U .$$

In the paper [2], F. Ronning introduced the class of univalent functions denoted by SP. We say that the function $f \in S$ is in SP if and only if

$$\mathbf{Re}\frac{zf'(z)}{f(z)} > \left|\frac{zf'(z)}{f(z)} - 1\right|, \tag{1}$$

for all $z \in U$.

The geometric interpretation of the relation (1) is that the class SP is the class of all functions $f \in S$ for which the expression $zf'(z)/f(z), z \in U$ takes all values in the parabolic region

$$\Omega = \{\omega : |\omega - 1| \leq \mathbf{Re}\omega\} = \left\{\omega = u + iv : v^2 \leq 2u - 1\right\}.$$

We consider the integral operator defined in [1]

$$F(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\alpha_1} \cdot \ldots \cdot \left(\frac{f_n(t)}{t}\right)^{\alpha_n} dt \tag{2}$$

and we study their properties.

Remark. We observe that for $n = 1$ and $\alpha_1 = 1$ we obtain the integral operator of Alexander.

2. Main results

Theorem 1.*Let $\alpha_i, i \in \{1, ..., n\}$ be the real numbers with the properties $\alpha_i > 0$ for $i \in \{1, ..., n\}$,and*

$$\sum_{i=1}^{n} \alpha_i \leq 1. \tag{3}$$

We suppose that the functions $f_i \in SP$ for $i = \{1, ..., n\}$. In this conditions the integral operator defined in (2) is convex of order $1 - \sum_{i=1}^{n} \alpha_i$.

Proof. We calculate for F the derivatives of the first and second order.

From (2) we obtain:

$$F'(z) = \left(\frac{f_1(z)}{z}\right)^{\alpha_1} \cdot \ldots \cdot \left(\frac{f_n(z)}{z}\right)^{\alpha_n}$$

and

$$F''(z)=\sum_{i=1}^{n}\alpha_i\left(\frac{f_i(z)}{z}\right)^{\alpha_i-1}\left(\frac{zf_i'(z)-f_i(z)}{zf_i(z)}\right)\prod_{\substack{j=1\\ j\neq i}}^{n}\left(\frac{f_j(z)}{z}\right)^{\alpha_j}$$

$$\frac{F''(z)}{F'(z)}=\alpha_1\left(\frac{zf_1'(z)-f_1(z)}{zf_1(z)}\right)+...+\alpha_n\left(\frac{zf_n'(z)-f_n(z)}{zf_n(z)}\right).$$

$$\frac{F''(z)}{F'(z)}=\alpha_1\left(\frac{f_1'(z)}{f_1(z)}-\frac{1}{z}\right)+...+\alpha_n\left(\frac{f_n'(z)}{f_n(z)}-\frac{1}{z}\right). \tag{4}$$

Multiply the relation (4) with z we obtain:

$$\frac{zF''(z)}{F'(z)}=\sum_{i=1}^{n}\alpha_i\left(\frac{zf_i'(z)}{f_i(z)}-1\right)=\sum_{i=1}^{n}\alpha_i\frac{zf_i'(z)}{f_i(z)}-\sum_{i=1}^{n}\alpha_i. \tag{5}$$

The relation (5) is equivalent with

$$\frac{zF''(z)}{F'(z)}+1=\sum_{i=1}^{n}\alpha_i\frac{zf_i'(z)}{f_i(z)}-\sum_{i=1}^{n}\alpha_i+1. \tag{6}$$

We calculate the real part from both terms of the above equality and obtain:

$$\mathbf{Re}\left(\frac{zF''(z)}{F'(z)}+1\right)=\sum_{i=1}^{n}\alpha_i\mathbf{Re}\left(\frac{zf_i'(z)}{f_i(z)}\right)-\sum_{i=1}^{n}\alpha_i+1. \tag{7}$$

Because $f_i\in SP$ for $i=\{1,...,n\}$ we apply in the above relation the inequality (1) and obtain:

$$\mathbf{Re}\left(\frac{zF''(z)}{F'(z)}+1\right)>\sum_{i=1}^{n}\alpha_i\left|\frac{zf_i'(z)}{f_i(z)}-1\right|-\sum_{i=1}^{n}\alpha_i+1. \tag{8}$$

Because $\alpha_i\left|\frac{zf_i'(z)}{f_i(z)}-1\right|>0$ for all $i\in\{1,...,n\}$, obtain that

$$\mathbf{Re}\left(\frac{zF''(z)}{F'(z)}+1\right)>1-\sum_{i=1}^{n}\alpha_i. \tag{9}$$

Using the hypothesis (3) in (9), we obtain that F is convex function of order $1-\sum_{i=1}^{n}\alpha_i$.

Remark. If $\sum_{i=1}^{n} \alpha_i = 1$ then

$$\mathbf{Re}\left(\frac{zF''(z)}{F'(z)}+1\right)>0 \tag{10}$$

so, F is the convex function.

Corollary 2. *Let γ be the real numbers with the properties $0<\gamma<1$. We suppose that the functions $f \in SP$. In this conditions the integral operator $F(z)=\int_0^z \left(\frac{f(z)}{z}\right)^{\gamma} dt$ is convex of order $1-\gamma$.*

Proof. In the Theorem 1, we consider $n=1$, $\alpha_1=\gamma$ and $f_1=f$.

Theorem 3. *We suppose that the function $f \in SP$. In this condition the integral operator of Alexander defined by*

$$F_A(z)=\int_0^z \frac{f(t)}{t}dt \tag{11}$$

is convex.

Proof. We have:

$$F'_A(z)=\frac{f(z)}{z}, F''_A(z)=\frac{zf'(z)-f(z)}{zf(z)}$$

and

$$\frac{zF''_A(z)}{F'_A(z)}=\frac{zf'(z)}{f(z)}-1. \tag{12}$$

From (12) we have:

$$\mathbf{Re}\left(\frac{zF''_A(z)}{F'_A(z)}+1\right)=\mathbf{Re}\frac{zf'(z)}{f(z)}>\left|\frac{zf'(z)}{f(z)}-1\right|>0. \tag{13}$$

So, the relation (13) imply that the Alexander operator F_A is convex.

Remark. Theorem 3 can be obtained from the Corollary 2 for $\gamma=1$.

References

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