C_0-SPACES AND C_1-SPACES

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Abstract. The aim of this paper is to introduce the concepts of C_0-spaces and C_1-spaces and study its basic properties in closure spaces.

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1. Introduction

Closure spaces were introduced by E. Čech [2] and then studied by many authors, see e.g. [3,4,5,6]. M. Caldas and S. Jafari [1] introduced the notions of $\wedge_\delta - R_0$ and $\wedge_\delta - R_1$ topological spaces as a modification of the known notions of R_0 and R_1 topological spaces. In this paper, we introduce the concepts of C_0 -spaces and C_1-spaces and study its basic properties in closure spaces.

2. Preliminaries

A map $u : P(X) \to P(X)$ defined on the power set $P(X)$ of a set X is called a *closure operator* on X and the pair (X, u) is called a *closure space* if the following axioms are satisfied :

(N1) $u\emptyset = \emptyset$,

(N2) $A \subseteq uA$ for every $A \subseteq X$,

(N3) $A \subseteq B \Rightarrow uA \subseteq uB$ for all $A, B \subseteq X$.

A closure operator u on a set X is called *additive* (respectively,*idempotent*) if $A, B \subseteq X \Rightarrow u(A \cup B) = uA \cup uB$ (respectively, $A \subseteq X \Rightarrow uuA = uA$). A subset $A \subseteq X$ is *closed* in the closure space (X, u) if $uA = A$ and it is *open* if its complement is closed. The empty set and the whole space are both open and closed.

A closure space (Y, v) is said to be a *subspace* of (X, u) if $Y \subseteq X$ and $vA = uA \cap Y$ for each subset $A \subseteq Y$. If Y is closed in (X, u), then the subspace (Y, v) of (X, u) is said to be closed too.

Let (X, u) and (Y, v) be closure spaces. A map $f : (X, u) \to (Y, v)$ is said to be *continuous* if $f(uA) \subseteq vf(A)$ for every subset $A \subseteq X$.

One can see that a map $f : (X, u) \to (Y, v)$ is continuous if and only if $uf^{-1}(B) \subseteq f^{-1}(vB)$ for every subset $B \subseteq Y$. Clearly, if $f : (X, u) \to (Y, v)$ is continuous, then $f^{-1}(F)$ is a closed subset of (X, u) for every closed subset F of (Y, v).

Let (X, u) and (Y, v) be closure spaces and let $f : (X, u) \to (Y, v)$ be a map. If f is continuous, then $f^{-1}(G)$ is an open subset of (X, u) for every open subset G of (Y, v).

Let (X, u) and (Y, v) be closure spaces. A map $f : (X, u) \to (Y, v)$ is said to be *closed* (resp. *open*) if $f(F)$ is a closed (resp. open) subset of (Y, v) whenever F is a closed (resp. open) subset of (X, u).

The *product* of a family $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ of closure spaces, denoted by $\prod_{\alpha \in I} (X_\alpha, u_\alpha)$, is the closure space $\big(\prod_{\alpha \in I} X_\alpha, u\big)$ where $\prod_{\alpha \in I} X_\alpha$ denotes the cartesian product of sets X_α, $\alpha \in I$, and u is the closure operator generated by the projections $\pi_\alpha : \prod_{\alpha \in I} (X_\alpha, u_\alpha) \to (X_\alpha, u_\alpha)$, $\alpha \in I$, i.e., is defined by $uA = \prod_{\alpha \in I} u_\alpha \pi_\alpha(A)$ for each $A \subseteq \prod_{\alpha \in I} X_\alpha$.

The following statement is evident:

Proposition 1. *Let $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ be a family of closure spaces and let $\beta \in I$. Then the projection map $\pi_\beta : \prod_{\alpha \in I} (X_\alpha, u_\alpha) \to (X_\beta, u_\beta)$ is closed and continuous.*

Proposition 2. *Let $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ be a family of closure spaces and let $\beta \in I$. Then F is a closed subset of (X_β, u_β) if and only if $F \times \prod_{\substack{\alpha \neq \beta \\ \alpha \in I}} X_\alpha$ is a closed subset of $\prod_{\alpha \in I} (X_\alpha, u_\alpha)$.*

Proof. Let $\beta \in I$ and let F be a closed subset of (X_β, u_β). Since π_β is continuous, $\pi_\beta^{-1}(F)$ is a closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$. But $\pi_\beta^{-1}(F) = F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$, hence $F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$.

Conversely, let $F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ be a closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$. Since π_β is closed, $\pi_\beta\Big(F \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha\Big) = F$ is a closed subset of (X_β, u_β). □

Proposition 3. *Let $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ be a family of closure spaces and let $\beta \in I$. Then G is an open subset of (X_β, u_β) if and only if $G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is an open subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$.*

Proof. Let $\beta \in I$ and let G be an open subset of (X_β, u_β). Since π_β is continuous, $\pi_\beta^{-1}(G)$ is an open subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$. But $\pi_\beta^{-1}(G) = G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$, therefore $G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is an open subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$.

Conversely, let $G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ be an open subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$. Then $\prod_{\alpha\in I} X_\alpha - G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$. But $\prod_{\alpha\in I} X_\alpha - G \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha = (X_\beta - G) \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$, hence $(X_\beta - G) \times \prod_{\substack{\alpha\neq\beta\\ \alpha\in I}} X_\alpha$ is a closed subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$. By Proposition 2, $X_\beta - G$ is a closed subset of (X_β, u_β). Consequently, G is an open subset of (X_β, u_β). □

3. C_0-Spaces and C_1-Spaces

Definition 4. A closure space (X, u) is said to be a C_0-*space* if, for every open subset G of (X, u), $x \in G$ implies $u\{x\} \subseteq G$.

Proposition 5. *A closure space (X, u) is a C_0 - space if and only if, for every closed subset F of (X, u) such that $x \notin F$, $u\{x\} \cap F = \emptyset$.*

Proof. Let F be a closed subset of (X, u) such that $x \notin F$. Then $X - F$ is an open subset of (X, u) such that $x \in X - F$. Since (X, u) is a C_0-space, $u\{x\} \subseteq X - F$. Consequently, $u\{x\} \cap F = \emptyset$.

Conversely, let G be an open subset of (X, u) and let $x \in G$. Then $X - G$ is a closed subset of (X, u) such that $x \notin X - G$. Therefore, $u\{x\} \cap (X - G) = \emptyset$. Consequently, $u\{x\} \subseteq G$. Hence, (X, u) is a C_0-space. □

Definition 6. A closure space (X, u) is said to be a C_1-*space* if, for each $x, y \in X$ such that $u\{x\} \neq u\{y\}$, there exists disjoint open subsets U and V of (X, u) such that $u\{x\} \subseteq U$ and $u\{y\} \subseteq V$.

Proposition 7. *Let (X, u) be a closure space. If (X, u) is a C_1-space, then (X, u) is a C_0-space.*

Proof. Let U be an open subset of (X, u) and let $x \in U$. If $y \notin U$, then $u\{x\} \neq u\{y\}$ because $x \notin u\{y\}$. Then there exists an open subset V_y of (X, u) such that $u\{y\} \subseteq V_y$ and $x \notin V_y$, which implies $y \notin u\{x\}$. Thus, $u\{x\} \subseteq U$. Hence, (X, u) is a C_0-space. □

The converse is not true as can be seen from the following example.

Example 8. *Let $X = \{a, b, c\}$ and define a closure operator $u : P(X) \to P(X)$ on X by $u\emptyset = \emptyset$, $u\{a\} = \{a\}$, $u\{b\} = u\{c\} = \{b, c\}$ and $u\{a, b\} = u\{a, c\} = u\{b, c\} = uX = X$. Then (X, u) is a C_0-space but not a C_1-space.*

Proposition 9. *A closure space (X, u) is a C_1-space if and only if, for every pair of points x, y of X such that $u\{x\} \neq u\{y\}$, there exists open subsets U and V of (X, u) such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.*

Proof. Suppose that (X, u) is a C_1-space. Let x, y be points of X such that $u\{x\} \neq u\{y\}$. There exists open subsets U and V of (X, u) such that $x \in u\{x\} \subseteq U$ and $y \in u\{y\} \subseteq V$.

Conversely, suppose that there exists open subsets U and V of (X, u) such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Since every C_1-space is C_0-space, $u\{x\} \subseteq U$ and $u\{y\} \subseteq V$. This gives the statement. □

Proposition 10. *Let $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ be a family of closure spaces. If $\prod_{\alpha \in I} (X_\alpha, u_\alpha)$ is a C_0-space, then (X_α, u_α) is a C_0-space for each $\alpha \in I$.*

Proof. Suppose that $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$ is a C_0-space. Let $\beta \in I$ and let G be an open subset of (X_β, u_β) such that $x_\beta \in G$. Then $G \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$ is an open subset of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$ such that $(x_\alpha)_{\alpha\in I} \in G \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$. Since $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$ is a C_0-space, $\prod_{\alpha\in I} u_\alpha\pi_\alpha(\{(x_\alpha)_{\alpha\in I}\}) \subseteq G \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$. Consequently, $u_\beta\{x_\beta\} \subseteq G$. Hence, (X_β, u_β) is a C_0-space. □

Proposition 11. *Let $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ be a family of closure spaces. If (X_α, u_α) is a C_1-space for each $\alpha \in I$, then $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$ is a C_1-space.*

Proof. Suppose that (X_α, u_α) is a C_1-space for each $\alpha \in I$. Let $(x_\alpha)_{\alpha\in I}$ and $(y_\alpha)_{\alpha\in I}$ be points of $\prod_{\alpha\in I} X_\alpha$ such that $\prod_{\alpha\in I} u_\alpha\pi_\alpha(\{(x_\alpha)_{\alpha\in I}\}) \neq \prod_{\alpha\in I} u_\alpha\pi_\alpha(\{(y_\alpha)_{\alpha\in I}\})$. There exist $\beta \in I$ such that $u_\beta\{x_\beta\} \neq u_\beta\{y_\beta\}$. Since (X_β, u_β) is a C_1-space, there exists open subsets U and V of (X_β, u_β) such that $u_\beta\{x_\beta\} \subseteq U$ and $u_\beta\{y_\beta\} \subseteq V$. Consequently,

$$\prod_{\alpha\in I} u_\alpha\pi_\alpha(\{(x_\alpha)_{\alpha\in I}\}) \subseteq U \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha,\ \prod_{\alpha\in I} u_\alpha\pi_\alpha(\{(y_\alpha)_{\alpha\in I}\}) \subseteq V \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$$

such that $U \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$ and $V \times \prod_{\substack{\alpha\neq\beta \\ \alpha\in I}} X_\alpha$ are open subsets of $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$. Hence, $\prod_{\alpha\in I}(X_\alpha, u_\alpha)$ is a C_1-space. □

References

[1] M. Caldas and S. Jafari, *On some low separation axioms in topological spaces*, Hoston Journal of Math., 29, (2003), 93-104.

[2] E. Čech, *Topological Spaces*, Topological Papers of Eduard Čech, Academia, Prague (1968), 436-472.

[3] J. Chvalina, *On homeomorphic topologies and equivalent set-systems*, Arch. Math. 2, Scripta Fac. Sci. Nat. UJEP Brunensis, XII, (1976), 107-116.

[4] J. Chvalina, *Stackbases in power sets of neighbourhood spaces preserving the continuity of mappings*, Arch. Math. 2, Scripta Fac. Sci. Nat. UJEP Brunensis, XVII, (1981), 81-86.

[5] L. Skula, *Systeme von stetigen abbildungen*, Czech. Math. J., 17, (92), (1967), 45-52.

[6] J. Šlapal, *Closure operations for digital topology*, Theoret. Comput. Sci., 305, (2003), 457-471.

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