ON THE GEOMETRY OF THE STANDARD k-SIMPLECTIC AND POISSON MANIFOLDS

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Abstract. The relation between the induced canonical connections on the reduced standard k-symplectic manifolds with respect to the action of a Lie group G is established. Similarly, defining Poisson brackets on these manifolds, the relation between the corresponding Poisson brackets on the reduced manifolds is stated.

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1. Introduction

The reduction is an important proceeder in symplectic mechanics. It has applications in fluids ([10]), electromagnetism and plasma physics ([9]), etc. Basicly, it consists of building new manifolds that inherit the same structures and similar properties as the initial manifolds. Applying the Marsden-Weinstein reduction for k-symplectic manifolds, we have shown ([5]) that a k-symplectic manifold gives by reduction a k-symplectic manifold, too. A particular case of k-symplectic manifold is the standard k-symplectic manifold $(T^1_k)^*R^n$ with the canonical k-symplectic structure induced from (R^n, ω_0) ([1]), that will be naturally identified with the Whitney sum of k copies of T^*R^n, that is $(T^1_k)^*R^n \equiv T^*R^n \oplus \overset{k}{\ldots} \oplus T^*R^n$ ([8]). Then, using a diffeomorphism, we can transfer on the k-tangent bundle $T^1_kR^n$ the standard k-symplectic structure from $(T^1_k)^*R^n$, that will be reduced, too. Similarly, $T^1_kR^n$ will be identified with the Whitney sum of k copies of TR^n. We proved that on a k-symplectic manifold, there exists a canonical connection ([7]). This canonical connection induces a canonical connection on the reduced manifold ([2]). Finally, we shall discuss the relation between the two induced canonical connections on the reduced standard k-symplectic manifolds. Similarly, a Poisson bracket on the standard k-symplectic manifolds shall be reduced and the relation between the two reduced Poisson brackets will be stated.

2. k-SYMPLECTIC STRUCTURES

Let M be an $(n+nk)$-dimensional smooth manifold.

Definition 1. ([1]) *We call* $(M,\omega_i,V)_{1\leq i\leq k}$ *k-symplectic manifold if* ω_i*,* $1\leq i\leq k$*, are k 2-forms and V is an nk-dimensional distribution that satisfy the conditions:*

1. ω_i *is closed, for every* $1\leq i\leq k$*;*
2. $\bigcap_{i=1}^{k}\ker\omega_i=\{0\}$*;*
3. $\omega_{i|_{V\times V}}=0$*, for every* $1\leq i\leq k$*.*

The canonical model for this structure is the k-cotangent bundle $(T_k^1)^*N$ of an arbitrary manifold N, which can be identified with the vector bundle $J^1(N,R^k)_0$ whose total space is the manifold of 1-jets of maps with target $0\in R^k$, and projection $\tau^*(j^1_{x,0}\sigma)=x$. We shall identify $(T_k^1)^*N$ with the Whitney sum of k copies of T^*N,

$$(T_k^1)^*N\cong T^*N\oplus \overset{k}{\ldots} \oplus T^*N,$$

$$j_{x,0}\sigma\mapsto (j^1_{x,0}\sigma^1,\ldots,j^k_{x,0}\sigma^k),$$

where $\sigma^i=\pi_i\circ\sigma : N\longrightarrow R$ is the i-th component of σ and the k-symplectic structure on $(T_k^1)^*N$ is given by

$$\omega_i=(\tau_i^*)^*(\omega_0)$$

and

$$V_{j^1_{x,0}\sigma}=\ker(\tau^*)_*(j^1_{x,0}\sigma),$$

where $\tau_i^* : (T_k^1)^*N\longrightarrow T^*N$ is the canonical projection on the i-th copy T^*N of $(T_k^1)^*N$ and ω_0 is the standard symplectic structure on T^*N.

3. THE STANDARD k-SYMPLECTIC MANIFOLDS

Let $\Phi : G\times R^n\to R^n$ be an action of a Lie group G on R^n. Define the lifted action $\Phi^{T_k^*} : G\times (T_k^1)^*R^n\to (T_k^1)^*R^n$ to the standard k-symplectic manifold $(T_k^1)^*R^n$:

$$\Phi^{T_k^*} : G\times (T_k^1)^*R^n\to (T_k^1)^*R^n,$$

$$\Phi^{T_k^*}(g, \alpha_{1q}, \dots, \alpha_{kq}) := (\alpha_{1q} \circ (\Phi_{g^{-1}})_{*\Phi_g(q)}, \dots, \alpha_{kq} \circ (\Phi_{g^{-1}})_{*\Phi_g(q)}), \tag{1}$$

$g \in G$, $(\alpha_1, \dots, \alpha_k) \in (T_k^1)^* R^n$, $q \in R^n$, which is a k-symplectic action ([11]), that is, it preserves the standard k-symplectic structure $\omega_1, \dots, \omega_k$ on $(T_k^1)^* R^n$. In a similar way, one can lift the action Φ to $T_k^1 R^n$:

$$\Phi^{T_k} : G \times T_k^1 R^n \to T_k^1 R^n,$$

$$\Phi^{T_k}(g, v_{1q}, \dots, v_{kq}) := ((\Phi_g)_{*q} v_{1q}, \dots, (\Phi_g)_{*q} v_{kq}), \tag{2}$$

$g \in G$, $(v_1, \dots, v_k) \in T_k^1 R^n$, $q \in R^n$.

Now, using a diffeomorphism $F : T_k^1 R^n \to (T_k^1)^* R^n$, equivariant with respect to the actions of G on $(T_k^1)^* R^n$ and $T_k^1 R^n$, we can take the pull-back on $T_k^1 R^n$ of the k-symplectic structure $(\omega_i, V)_{1 \le i \le k}$ on the standard k-symplectic manifold $(T_k^1)^* R^n$ ([8]), and define $((\omega_F)_i, V_F)_{1 \le i \le k}$ by:

1. $(\omega_F)_i = F^* \omega_i$,
2. $V_F = \ker(\pi_F)_*$,

for any $1 \le i \le k$, where $\pi_F : T_k^1 R^n \to R^n$, $\pi_F(v_{1q}, \dots, v_{kq}) := q$. Then $(T_k^1 R^n, (\omega_F)_i, V_F)_{1 \le i \le k}$ is a k-symplectic manifold and F becomes a symplectomorphism between $(T_k^1 R^n, (\omega_F)_i, V_F)_{1 \le i \le k}$ and $((T_k^1)^* R^n, \omega_i, V)_{1 \le i \le k}$. For instance, such a diffeomorphism between $T_k^1 R^n$ and $(T_k^1)^* R^n$ can be the Legendre transformation TL associated to a regular Lagrangian $L \in C^\infty(T_k^1 R^n, R)$, that is

$$TL : T_k^1 R^n \to (T_k^1)^* R^n$$

defined by

$$(TL(v_{1q}, \dots, v_{kq}))^i(w_q) := \frac{d}{ds} |_{s=0} L(v_{1q}, \dots, v_{iq} + s w_q, \dots, v_{kq}), \ (\forall) 1 \le i \le k. \tag{3}$$

4. Canonical connections and Poisson structures

If the Lie group G acts freely and properly on $T_k^1 R^n$ and $(T_k^1)^* R^n$, then the quotient spaces $T_k^1 R^n / G$ and $(T_k^1)^* R^n / G$ are smooth manifolds. We proved that on any k-symplectic manifold, there exists a canonical connection ([7]). On the two standard k-symplectic manifolds described above, consider the two canonical connections ∇ on $(T_k^1)^*R^n$ and $\bar{\nabla}$ on $T_k^1R^n$ which induce, naturally, on the reduced manifolds $(T_k^1)^*R^n/G$ and $T_k^1R^n/G$ respectively the reduced canonical connections ∇^G and $\bar{\nabla}^G$ ([2]).

As F is compatible with the equivalence relations that define the quotient manifolds $T_k^1R^n/G$ and $(T_k^1)^*R^n/G$, it induces a diffeomorphism $[F]$: $T_k^1R^n/G \to (T_k^1)^*R^n/G$ such that the following diagram commutes:

$$\begin{array}{ccc} T_k^1R^n & \xrightarrow{F} & (T_k^1)^*R^n \\ \pi^{T_k}\downarrow & & \downarrow \pi^{T_k^*} \\ T_k^1R^n/G & \xrightarrow{[F]} & (T_k^1)^*R^n/G \end{array}$$

where $\pi^{T_k^*} : (T_k^1)^*R^n \to (T_k^1)^*R^n/G$ and $\pi^{T_k} : T_k^1R^n \to T_k^1R^n/G$ are the canonical projections.

Then we have

Proposition 1. ([3]) *The two reduced connections are connected by the relation*

$$[F]_* \circ \bar{\nabla}^G = \nabla^G \circ ([F]_* \times [F]_*). \tag{4}$$

Consider now a Hamiltonian $H \in C^\infty((T_k^1)^*R^n, R)$ and denote by $X_H^i = (X_{1H}^i, \dots, X_{kH}^i)$, $1 \le i \le k$, the *Hamiltonian vector fields* on $(T_k^1)^*R^n$ associated to H ([13]).

Proposition 2. ([13]) *A Poisson bracket on* $(T_k^1)^*R^n$ *is given by*

$$\{f, h\} = \sum_{i=1}^{k} \omega_i(X_{if}^i, X_{ih}^i), \tag{5}$$

where $f, h \in C^\infty((T_k^1)^*R^n, R)$ *and* $X_f^i = (X_{1f}^i, \dots, X_{kf}^i)$, $X_h^i = (X_{1h}^i, \dots, X_{kh}^i)$, $1 \le i \le k$, *are the corresponding Hamiltonian vector fields.*

In ([6]) we have proved that

$$\{f, h\}_F = F^*\{F^{*-1}f, F^{*-1}h\}, \tag{6}$$

$f, h \in C^\infty(T_k^1R^n, R)$, is a Poisson bracket on $T_k^1R^n$.

We want to induce Poisson brackets on $T_k^1R^n/G$ and $(T_k^1)^*R^n/G$. For that, we need some additional assumptions concerning the actions of G on these spaces.

Assume that G acts canonically on $T_k^1R^n$ and $(T_k^1)^*R^n$ via the maps $\Phi_g^{T_k^*}$ and $\Phi_g^{T_k}$ respectively, that is

$$(\Phi_g^{T_k})^*\{f,h\}_F = \{(\Phi_g^{T_k})^*(f), (\Phi_g^{T_k})^*(h)\}_F, \quad (\forall) g \in G,$$

$f, h \in C^\infty(T_k^1R^n, R)$ and

$$(\Phi_g^{T_k^*})^*\{f,h\} = \{(\Phi_g^{T_k^*})^*(f), (\Phi_g^{T_k^*})^*(h)\}, \quad (\forall) g \in G,$$

$f, h \in C^\infty((T_k^1)^*R^n, R)$.

In this case, following ([12]), the reduced spaces $(T_k^1)^*R^n/G$ and $T_k^1R^n/G$ are Poisson manifolds, too, with the Poisson brackets given by

$$\{f,h\}^{(T_k^1)^*R^n/G}(\pi^{T_k^*}(\alpha_1,\ldots,\alpha_k)) := \{f\circ\pi^{T_k^*}, h\circ\pi^{T_k^*}\}(\alpha_1,\ldots,\alpha_k), \qquad (7)$$

$f, h \in C^\infty((T_k^1)^*R^n/G, R)$, $(\alpha_1,\ldots,\alpha_k) \in (T_k^1)^*R^n$ and respectively by

$$\{f,h\}_F^{T_k^1R^n/G}(\pi^{T_k}(v_1,\ldots,v_k)) := \{f\circ\pi^{T_k}, h\circ\pi^{T_k}\}_F(v_1,\ldots,v_k), \qquad (8)$$

$f, h \in C^\infty(T_k^1R^n/G, R)$, $(v_1,\ldots,v_k) \in T_k^1R^n$.

Then we have

Proposition 3. ([4]) *For any $f, h \in C^\infty((T_k^1)^*R^n/G, R)$, the two reduced Poisson brackets are connected by the relation*

$$[F]^*\{f,h\}^{(T_k^1)^*R^n/G} = \{[F]^*(f), [F]^*(h)\}_F^{T_k^1R^n/G}. \qquad (9)$$

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