A CLASS OF MULTIVALENT ANALYTIC FUNCTIONS INVOLVING THE GENERALIZED JUNG-KIM-SRIVASTAVA OPERATOR

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Abstract. By making use of subordination between analytic functions and the generalized Jung-Kim-Srivastava operator, we introduce and investigate a certain subclass of p-valent analytic functions. Such results as inclusion relationship, subordination property, integral preserving property and argument estimate are proved.

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1. Introduction

In 2006, Shams *et al.* [12] introduced and investigated the following two-parameter family of integral operators:

$$\mathcal{I}_p^{\delta} f(z) := \frac{(p+1)^{\delta}}{z\Gamma(\delta)} \int_0^z \left(\log \frac{z}{t}\right)^{\delta-1} f(t)dt \qquad (\delta > 0) \tag{1}$$

for the functions $f \in \mathcal{A}(p)$, where $\mathcal{A}(p)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}, \tag{2}$$

which are *analytic* in the *open* unit disk

$$U := \{z : \ z \in C \quad \text{and} \quad |z| < 1\}.$$

We note that $\mathcal{I}_1^{\delta}$ is the well-known Jung-Kim-Srivastava operator [3]. In recent years, Li [4] , Liu [5,6] and Uralogaddi and Somanatha [13] obtained many interesting results associated with the Jung-Kim-Srivastava operator.

It is readily verified from (1) that

$$\mathcal{I}_p^\delta f(z) = z^p + \sum_{n=1}^{\infty} \left(\frac{p+1}{n+p+1}\right)^\delta a_{p+n} z^{p+n}, \tag{3}$$

and

$$z\left(\mathcal{I}_p^\delta f(z)\right)' = (p+1)\mathcal{I}_p^{\delta-1} f(z) - \mathcal{I}_p^\delta f(z). \tag{4}$$

Let $\mathcal{P}$ denote the class of functions of the form:

$$p\,(z) = 1 + \sum_{n=1}^{\infty} p_n z^n,$$

which are analytic and convex in U and satisfy the condition:

$$\Re(p\,(z)) > 0 \qquad (z \in U).$$

For two functions f and g, analytic in U, we say that the function f is subordinate to g in U, and write

$$f(z) \prec g(z) \quad \text{or} \quad f \prec g,$$

if there exists a Schwarz function ω, which is analytic in U with

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \quad (z \in U)$$

such that

$$f(z) = g(\omega(z)) \qquad (z \in U).$$

By making use of the operator $\mathcal{I}_p^\delta$ and the above-mentioned principle of subordination between analytic functions, we introduce and investigate the following subclass of p-valent functions.

Definition 1 *A function $f \in \mathcal{A}(p)$ is said to be in the class $\mathcal{S}_p^\delta(\eta; h)$ if it satisfies the following differential subordination:*

$$\frac{1}{p-\eta}\left(\frac{z\left(\mathcal{I}_p^\delta f(z)\right)'}{\mathcal{I}_p^\delta f(z)} - \eta\right) \prec h(z) \qquad (z \in U;\ 0 \le \eta < p\,;\ h \in \mathcal{P}). \tag{5}$$

For simplicity, we write

$$\mathcal{S}_p^\delta\left(\eta; \frac{1+Az}{1+Bz}\right) =: \mathcal{S}_p^\delta(\eta; A, B) \qquad (-1 \leq B < A \leq 1).$$

The family $\mathcal{S}_p^\delta(\eta; h)$ is a comprehensive family containing various well-known as well as new classes of analytic functions. For example, for $\delta = 0$, we get the class $\mathcal{S}_p^*(\eta; h)$ studied by Cho *et al.* [1], in case of $\delta = 0,\ A = 1$ and $B = -1$, we get the class $\mathcal{S}_p^*(\eta)$ consisting of all p-valent starlike functions of order η.

In the present paper, we aim at proving such results as inclusion relationship, subordination property, integral preserving property and argument estimate for the class $\mathcal{S}_p^\delta(\eta; h)$.

2. Preliminary Results

In order to prove our main results, we need the following lemmas.

Lemma 1 (See [2]) *Let $\zeta, \vartheta \in C$. Suppose that m is convex and univalent in U with*

$$m(0) = 1 \quad \textit{and} \quad \Re(\zeta m(z) + \vartheta) > 0 \qquad (z \in U).$$

If u is analytic in U with $u(0) = 1$, then the following subordination:

$$u(z) + \frac{zu'(z)}{\zeta u(z) + \vartheta} \prec m(z) \qquad (z \in U)$$

implies that

$$u(z) \prec m(z) \qquad (z \in U).$$

Lemma 2 (See [7]) *Let h be convex univalent in U and w be analytic in U with*

$$\Re(w(z)) \geq 0 \qquad (z \in U).$$

If q is analytic in U and $q(0) = h(0)$, then the subordination

$$q(z) + w(z)zq'(z) \prec h(z) \qquad (z \in U)$$

implies that

$$q(z) \prec h(z) \qquad (z \in U).$$

Lemma 3 (See [9]) *Let q be analytic in U with $q(0) = 1$ and $q(z) \neq 0$ for all $z \in U$. If there exist two points $z_1, z_2 \in U$ such that*

$$-\frac{\pi}{2}\alpha_1 = \arg(q(z_1)) < \arg(q(z)) < \arg(q(z_2)) = \frac{\pi}{2}\alpha_2,$$

for some α_1 and α_2 ($\alpha_1,\ \alpha_2 > 0$) and for all z ($|z| < |z_1| = |z_2|$), then

$$\frac{z_1 q'(z_1)}{q(z_1)} = -i\left(\frac{\alpha_1 + \alpha_2}{2}\right)m$$

and

$$\frac{z_2 q'(z_2)}{q(z_2)} = i\left(\frac{\alpha_1 + \alpha_2}{2}\right)m$$

where

$$m \geq \frac{1 - |b|}{1 + |b|} \quad \text{and} \quad b = i\tan\frac{\pi}{4}\left(\frac{\alpha_2 - \alpha_1}{\alpha_1 + \alpha_2}\right).$$

Lemma 4 (See [11]) *The function*

$$(1 - z)^{\gamma} \equiv \exp(\gamma \log(1 - z)) \qquad (\gamma \neq 0)$$

is univalent if and only if γ is either in the closed disk $|\gamma - 1| \leq 1$ or in the closed disk $|\gamma + 1| \leq 1$.

Lemma 5 (See [8]) *Let $q(z)$ be univalent in U and let $\theta(w)$ and $\varphi(w)$ be analytic in a domain D containing $q(U)$ with $\varphi(w) \neq 0$ when $w \in q(U)$. Set*

$$Q(z) = zq'(z)\varphi(q(z)), \quad h(z) = \theta(q(z)) + Q(z)$$

and suppose that

1. *$Q(z)$ is starlike (univalent) in U;*
2. *$\Re\left(\frac{zh'(z)}{Q(z)}\right) = \Re\left(\frac{\theta'(q(z))}{\varphi(q(z))} + \frac{zQ'(z)}{Q(z)}\right) > 0 \quad (z \in U)$.*

If p is analytic in U with $p(0) = q(0)$ and $p(U) \subset D$, and

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z)) = h(z),$$

then $p(z) \prec q(z)$, and q is the best dominant.

3. Main Results

We begin by presenting the following inclusion relationship for the class $\mathcal{S}_p^{\delta}(\eta; h)$.

Theorem 1 *Let* $f \in \mathcal{S}_p^{\delta-1}(\eta; h)$ *with*

$$\Re((p-\eta)h(z) + \eta + 1) > 0.$$

Then

$$\mathcal{S}_p^{\delta-1}(\eta; h) \subset \mathcal{S}_p^{\delta}(\eta; h).$$

Proof. Let $f \in \mathcal{S}_p^{\delta-1}(\eta; h)$ and suppose that

$$q(z) := \frac{1}{p-\eta}\left(\frac{z\left(\mathcal{I}_p^{\delta} f(z)\right)'}{\mathcal{I}_p^{\delta} f(z)} - \eta\right). \tag{6}$$

Then q is analytic in U with $q(0) = 1$. Combining (4) and (6), we obtain

$$(p+1)\frac{\mathcal{I}_p^{\delta-1} f(z)}{\mathcal{I}_p^{\delta} f(z)} = (p-\eta)q(z) + \eta + 1. \tag{7}$$

By logarithmically differentiating both sides of (7) and using (6), we get

$$\frac{1}{p-\eta}\left(\frac{z\left(\mathcal{I}^{\delta-1} f(z)\right)'}{\mathcal{I}_p^{\delta-1} f(z)} - \eta\right) = q(z) + \frac{zq'(z)}{(p-\eta)q(z) + \eta + 1} \prec h(z). \tag{8}$$

Since

$$\Re((p-\eta)h(z) + \eta + 1) > 0,$$

an application of Lemma 1 to (8) yields

$$q(z) \prec h(z) \qquad (z \in U),$$

which implies that the assertion of Theorem 1 holds.

Theorem 2 *Let* $1 < \rho < 2$ *and* $\gamma \neq 0$ *be a real number satisfying either*

$$|2\gamma(\rho - 1)(p + 1) - 1| \leq 1$$

or

$$|2\gamma(\rho - 1)(p + 1) + 1| \leq 1.$$

If $f \in \mathcal{A}(p)$ *satisfies the condition*

$$\Re\left(1 + \frac{\mathcal{I}_p^{\delta-1} f(z)}{\mathcal{I}_p^{\delta} f(z)}\right) > 2 - \rho \qquad (z \in U), \tag{9}$$

then

$$\left(z\mathcal{I}_p^{\delta} f(z)\right)^{\gamma} \prec q_1(z) = \frac{1}{(1-z)^{2\gamma(\rho-1)(p+1)}}, \tag{10}$$

where q_1 *is the best dominant.*

Proof. Suppose that

$$p(z) := \left(z\mathcal{I}_p^{\delta} f(z)\right)^{\gamma}.$$

It follows that

$$\frac{zp'(z)}{p(z)} = \gamma(p+1)\frac{\mathcal{I}_p^{\delta-1} f(z)}{\mathcal{I}_p^{\delta} f(z)}. \tag{11}$$

Combining (9) and (11), we find that

$$1 + \frac{zp'(z)}{\gamma(p+1)p(z)} \prec \frac{1 + (2\rho - 3)z}{1 - z}. \tag{12}$$

If we choose

$$\theta(w) = 1, \quad q_1(z) = \frac{1}{(1-z)^{2\gamma(\rho-1)(p+1)}} \quad \text{and} \quad \varphi(w) = \frac{1}{\gamma w(p+1)},$$

then by the assumption of theorem and making use of Lemma 4, we know that q_1 is univalent in U. It now follows that

$$Q(z) = zq'(z)\varphi(q(z)) = \frac{2(\rho - 1)z}{1 - z},$$

and

$$\theta(q(z)) + Q(z) = \frac{1 + (2\rho - 3)z}{1 - z} = h(z).$$

If we define D by

$$q(U) = \left\{\omega : \ \left|\omega^{\frac{1}{\zeta}} - 1\right| < \left|\omega^{\frac{1}{\zeta}}\right| \quad (\zeta = 2\gamma(p-1)(p+1))\right\} \subset D,$$

then, it is easy to check that the conditions (1) and (2) of Lemma 5 hold true. Thus, the desired result of Theorem 2 follows from (12).

Theorem 3 *Let* $f \in \mathcal{S}_p^{\delta}(\eta; h)$ *with*

$$\Re((p-\eta)h(z) + \mu + \eta) > 0 \qquad (z \in U).$$

Then the integral operator F *defined by*

$$F(z) = \frac{\mu + p}{z^{\mu}} \int_0^z t^{\mu - 1} f(t) dt \qquad (z \in U) \tag{13}$$

belongs to the class $\mathcal{S}_p^{\delta}(\eta, h)$.

Proof. Let $f \in \mathcal{S}_p^{\delta}(\eta; h)$. Then, from (13), we find that

$$z\left(\mathcal{I}_p^{\delta} F(z)\right)' + \mu \mathcal{I}_p^{\delta} F(z) = (\mu + p)\mathcal{I}_p^{\delta} f(z). \tag{14}$$

By setting

$$q_2(z) := \frac{1}{p-\eta}\left(\frac{z\left(\mathcal{I}_p^{\delta} F(z)\right)'}{\mathcal{I}_p^{\delta} F(z)} - \eta\right). \tag{15}$$

we observe that q_2 is analytic in U with $q_2(0) = 0$. It follows from (14) and (15) that

$$(\mu + p)\frac{\mathcal{I}_p^{\delta} f(z)}{\mathcal{I}_p^{\delta} F(z)} = \mu + \eta + (p-\eta) q_2(z). \tag{16}$$

Differentiating both sides of (16) with respect to z logarithmically and using (15), we get

$$q_2(z) + \frac{z q_2'(z)}{\mu + \eta + (p-\eta) q_2(z)} = \frac{1}{p-\eta}\left(\frac{z\left(\mathcal{I}_p^{\delta} f(z)\right)'}{\mathcal{I}_p^{\delta} f(z)} - \eta\right) \prec h(z) \qquad (z \in U). \tag{17}$$

Since

$$\Re((p-\eta)h(z) + \mu + \eta) > 0 \qquad (z \in U).$$

An application of Lemma 1 to (17) yields

$$\frac{1}{p-\eta}\left(\frac{z\left(\mathcal{I}_p^{\delta}F(z)\right)'}{\mathcal{I}_p^{\delta}F(z)}-\eta\right)\prec h(z),$$

which implies that the assertion of Theorem 3 holds.

Theorem 4 *Let* $f\in\mathcal{A}(p),\ 0<\delta_1,\ \delta_2\leq 1$ *and* $0\leq\eta<p$. *If*

$$-\frac{\pi}{2}\delta_1<\arg\left(\frac{z\left(\mathcal{I}_p^{\delta-1}f(z)\right)'}{\mathcal{I}_p^{\delta-1}g(z)}-\eta\right)<\frac{\pi}{2}\delta_2,$$

for some $g\in\mathcal{S}_p^{\delta-1}(\eta;A,B)$, *then*

$$-\frac{\pi}{2}\alpha_1<\arg\left(\frac{z\left(\mathcal{I}_p^{\delta}f(z)\right)'}{\mathcal{I}_p^{\delta}g(z)}-\eta\right)<\frac{\pi}{2}\alpha_2,$$

where α_1 *and* α_2 $(0<\alpha_1,\ \alpha_2\leq 1)$ *are the solutions of the following equations*

$$\delta_1=\begin{cases}\alpha_1+\frac{2}{\pi}\tan^{-1}\left(\frac{(1-|b|)(\alpha_1+\alpha_2)\cos\frac{\pi}{2}t}{2(1+|b|)(\frac{(p-\eta)(1+A)}{1+B}+\eta+1)+(1-|b|)(\alpha_1+\alpha_2)\sin\frac{\pi}{2}t}\right) & (B\neq -1),\\ \alpha_1 & (B=-1),\end{cases}$$

and

$$\delta_2=\begin{cases}\alpha_2+\frac{2}{\pi}\tan^{-1}\left(\frac{(1-|b|)(\alpha_1+\alpha_2)\cos\frac{\pi}{2}t}{2(1+|b|)(\frac{(p-\eta)(1+A)}{1+B}+\eta+1)+(1-|b|)(\alpha_1+\alpha_2)\sin\frac{\pi}{2}t}\right) & (B\neq -1),\\ \alpha_2 & (B=-1),\end{cases}$$

with

$$b=\tan\frac{\pi}{4}\left(\frac{\alpha_2-\alpha_1}{\alpha_2+\alpha_1}\right),$$

and

$$t=\frac{2}{\pi}\sin^{-1}\left(\frac{(p-\eta)(A-B)}{(p-\eta)(1-AB)+(\eta+1)(1-B^2)}\right).$$

Proof. Suppose that

$$q_3(z) := \frac{1}{p-\gamma}\left(\frac{z\left(\mathcal{I}_p^{\delta} f(z)\right)'}{\mathcal{I}_p^{\delta} g(z)} - \gamma\right) \tag{18}$$

with $0 \leq \gamma < p$ and $g \in \mathcal{S}_p^{\delta-1}(\eta; A, B)$. Then $q_3(z)$ is analytic in U with $q_3(0) = 1$. It follows from (4) and (18) that

$$[(p-\gamma)q_3(z) + \gamma]\mathcal{I}_p^{\delta} g(z) = (p+1)\mathcal{I}_p^{\delta-1} f(z) - \mathcal{I}_p^{\delta} f(z). \tag{19}$$

Differentiating both sides of (19) and multiplying the resulting equation by z, we get

$$(p-\gamma)zq_3'(z)\mathcal{I}_p^{\delta} g(z) + [(p-\gamma)q_3(z)+\gamma]z\left(\mathcal{I}_p^{\delta} g(z)\right)' = (p+1)z\left(\mathcal{I}_p^{\delta-1} f(z)\right)' - z\left(\mathcal{I}_p^{\delta} f(z)\right)'. \tag{20}$$

Since $g \in \mathcal{S}_p^{\delta-1}(\eta; A, B)$, by Theorem 1, we know that $g \in \mathcal{S}_p^{\delta}(\eta; A, B)$. If we set

$$q_4(z) := \frac{1}{p-\eta}\left(\frac{z\left(\mathcal{I}_p^{\delta} g(z)\right)'}{\mathcal{I}_p^{\delta} g(z)} - \eta\right), \tag{21}$$

combining (4) (with f replaced by g) and (21), we easily get

$$\frac{\mathcal{I}_p^{\delta} g(z)}{\mathcal{I}_p^{\delta-1} g(z)} = \frac{p+1}{(p-\eta)q_4(z) + \eta + 1}. \tag{22}$$

Now, from (18), (21) and (22), we find that

$$\frac{1}{p-\gamma}\left(\frac{z\left(\mathcal{I}_p^{\delta-1} f(z)\right)'}{\mathcal{I}_p^{\delta-1} g(z)} - \gamma\right) = q_3(z) + \frac{zq_3'(z)}{(p-\eta)q_4(z) + \eta + 1}. \tag{23}$$

Since

$$q_4(z) \prec \frac{1+Az}{1+Bz} \qquad (-1 \leq B < A \leq 1),$$

it is easy to see that

$$\left|q_4(z) - \frac{1-AB}{1-B^2}\right| < \frac{A-B}{1-B^2} \qquad (z \in U;\ B \neq -1) \tag{24}$$

and

$$\Re(q_4(z)) > \frac{1-A}{2} \qquad (z \in U;\ B=-1). \tag{25}$$

We now easily find from (24) and (25) that

$$\left|(p-\eta)q_4(z)+\eta+1-\frac{(\eta+1)(1-B^2)+(p-\eta)(1-AB)}{1-B^2}\right| < \frac{(p-\eta)(A-B)}{1-B^2}$$

$$(B \neq -1),$$

and

$$\Re\left((p-\eta)q_4(z)+\eta+1\right) > \frac{(1-A)(p-\eta)}{2}+\eta+1 \qquad (B=-1).$$

If we set

$$(p-\eta)q_4(z)+\eta+1 = r\exp\left(i\frac{\pi}{2}\theta\right),$$

where

$$-\rho<\theta<\rho \qquad \left(\rho := \frac{(p-\eta)(A-B)}{(p-\eta)(1-AB)+(\eta+1)(1-B^2)}\right) \qquad (B\neq -1)$$

and

$$-1<\theta<1 \qquad (B=-1),$$

then

$$\frac{(p-\eta)(1-A)}{1-B}+\eta+1<r<\frac{(p-\eta)(1+A)}{1+B}+\eta+1 \qquad (B\neq -1)$$

and

$$\frac{(p-\eta)(1-A)}{2}+\eta+1<r \qquad (B=-1),$$

Since q_3 is analytic in U with $q_3(0)=1$, an application of Lemma 2 to (23) yields $q_3(z) \prec h(z)$.

Next, we suppose that

$$Q(z) = \frac{1}{p-\gamma}\left(\frac{z\left(\mathcal{I}_p^{\delta-1}f(z)\right)'}{\mathcal{I}_p^{\delta-1}g(z)}-\gamma\right) \qquad (0\leq\gamma<p). \tag{26}$$

Combining (23) and (26), we get

$$\arg(Q(z)) = \arg(q_3(z)) + \arg\left(1 + \frac{zq_3'(z)}{[(p-\eta)q_4(z)+\eta+1]q_3(z)}\right).$$

If there exist two points $z_1, z_2 \in U$ such that

$$-\frac{\pi}{2}\alpha_1 = \arg(q_3(z_1)) < \arg(q_3(z)) < \arg(q_3(z_2)) = \frac{\pi}{2}\alpha_2,$$

by Lemma 3, we know that

$$\frac{z_1q_3'(z_1)}{q_3(z_1)} = -i\left(\frac{\alpha_1+\alpha_2}{2}\right)m \quad \text{and} \quad \frac{z_2q_3'(z_2)}{q_3(z_2)} = i\left(\frac{\alpha_1+\alpha_2}{2}\right)m$$

where

$$m \geq \frac{1-|b|}{1+|b|} \quad \text{and} \quad b = i\tan\frac{\pi}{4}\left(\frac{\alpha_2-\alpha_1}{\alpha_2+\alpha_1}\right).$$

The following we split it into two cases to prove.

1. When $B \neq -1$, we have

$$\begin{aligned}
&\arg(Q(z_1))\\
&= -\tfrac{\pi}{2}\alpha_1 + \arg\left(1 - im\left(\tfrac{\alpha_1+\alpha_2}{2}\right)r^{-1}e^{-i\frac{\pi}{2}\theta}\right)\\
&= -\tfrac{\pi}{2}\alpha_1 + \arg\left(1 - \tfrac{m}{2r}(\alpha_1+\alpha_2)\cos\tfrac{\pi}{2}(1-\theta) + \tfrac{im}{2r}(\alpha_1+\alpha_2)\sin\tfrac{\pi}{2}(1-\theta)\right)\\
&\leq -\tfrac{\pi}{2}\alpha_1 - \tan^{-1}\left(\tfrac{m(\alpha_1+\alpha_2)\sin\frac{\pi}{2}(1-\theta)}{2r+m(\alpha_1+\alpha_2)\cos\frac{\pi}{2}(1-\theta)}\right)\\
&\leq -\tfrac{\pi}{2}\alpha_1 - \tan^{-1}\left(\tfrac{(1-|b|)(\alpha_1+\alpha_2)\cos\frac{\pi}{2}t}{2(1+|b|)(\frac{(p-\eta)(1+A)}{1+B}+\eta+1)+(1-|b|)(\alpha_1+\alpha_2)\sin\frac{\pi}{2}t}\right)\\
&= -\tfrac{\pi}{2}\delta_1,
\end{aligned}$$

and

$$\begin{aligned}
&\arg(Q(z_2))\\
&= \arg(q_3(z_2)) + \arg\left(1 + \tfrac{z_2q_3'(z_2)}{[(p-\eta)q_4(z_2)+\eta+1]q_3(z_2)}\right)\\
&\geq \tfrac{\pi}{2}\alpha_2 + \tan^{-1}\left(\tfrac{(1-|b|)(\alpha_1+\alpha_2)\cos\frac{\pi}{2}t}{2(1+|b|)(\frac{(p-\eta)(1+A)}{1+B}+\eta+1)+(1-|b|)(\alpha_1+\alpha_2)\sin\frac{\pi}{2}t}\right)\\
&= \tfrac{\pi}{2}\delta_2.
\end{aligned}$$

2. For the case $B = -1$, we similarly obtain

$$\arg(Q(z_1)) = \arg\left(q_3(z_1) + \frac{z_1q_3'(z_1)}{(p-\eta)q_4(z_1)+\eta+1}\right) \leq -\frac{\pi}{2}\alpha_1,$$

and

$$\arg(Q(z_2)) = \arg\left(q_3(z_2) + \frac{z_2 q_3'(z_2)}{(p-\eta)q_4(z_2)+\eta+1}\right) \geq \frac{\pi}{2}\alpha_2.$$

The above two cases contradict the assumptions of Theorem 4. The proof of Theorem 4 is thus completed.

Remark. Generally, all bounds in Theorem 4 are not sharp (see, for details, [1] and [10]), the sharpness is still an open problem.

References

[1] N. E. Cho, O. S. Kwon and H. M. Srivastava, *Inclusion relationships and argument properties for certain subclasses of multivalent functions associated with a family of linear operators*, J. Math. Anal. Appl. **292** (2004), 470–483.

[2] P. Eenigenburg, S. S. Miller, P. T. Mocanu and M. O. Reade, *On a Briot-Bouquet differential subordination*, Rev. Roumaine Math. Pures Appl. **29** (1984), 567–573.

[3] I. B. Jung, Y. C. Kim, and H. M. Srivastava, *The Hardy space of analytic functions associated with certain one-parameter families of integral operators,* J. Math. Anal. Appl. **176** (1993), 138–147.

[4] J.-L. Li, *Some properties of two integral operators*, Soochow J. Math. **25** (1999), 91–96.

[5] J.-L. Liu, *On a class of p-valent analytic functions*, Chinese Quart. J. Math. **15** (2000), 27–32.

[6] J.-L. Liu, *A linear operator and strongly starlike functions.* J. Math. Soc. Japan **54** (2002), 975–981.

[7] S. S. Miller and P. T. Mocanou, *Differential subordinations and univalent functions,* Michigan Math. J. **28** (1981), 157–171.

[8] S. S. Miller and P. T. Mocanou, *On some classes of first order differential subordinations,* Michigan Math. J. **32** (1985), 185–195.

[9] M. Nunokawa, S. Owa, H. Saitoh, N. E. Cho and N. Takahashi, *Some properties of analytic functions at extermal points for arguments*, preprint.

[10] M. Nunokawa, S. Owa, H. Saitoh, A. Ikeda and N. Koike, *Some results for strongly starlike functions*, J. Math. Anal. Appl. **212** (1997), 98–106.

[11] M. S. Robertson, *Certain classes of starlike functions,* Michigan Math. J. **32** (1985), 135–140.

[12] S. Shams, S. R. Kulkarni and J. M. Jahangiri, *Subordination properties of p-valent functions defined by integral operators*, Int. J. Math. Math. Sci. Article ID 94572, (2006), pp. 1–3.

[13] B. A. Uralegaddi and C. Somantha, *Certain integral operators for starlike functions*, J. Math. Res. Exposition **15** (1995), 14–16.

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