ON NEW CLASSES OF UNIVALENT HARMONIC FUNCTIONS DEFINED BY GENERALIZED DIFFERENTIAL OPERATOR

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Abstract. In this article, we define two classes of univalent harmonic functions in the open unit disk

$$U := \{z \in \mathbb{C} : |z| < 1\}$$

under certain conditions involving generalized differential operator introduced by the first author [10, 11] as follows

$$\mathcal{D}^k_{\lambda,\delta} f(z) = z + \sum_{n=2}^{\infty} [1 + (n-1)\lambda]^k C(\delta, n) a_n z^n, \;\; k \in \mathbb{N}_0, \lambda \geq 0, \, \delta \geq 0, \quad (z \in U)$$

for analytic function of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in U)$. A sufficient coefficient, such as distortion bounds, extreme points and other properties are studied.

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1. Introduction

Let $U := \{z : |z| < 1\}$ be the open unit disk and let S_H denote the class of all complex valued, harmonic, sense-preserving, univalent functions f in U normalized by $f(0) = f'(0) - 1 = 0$ and expressed as $f(z) = h(z) + \overline{g(z)}$ where

h and g belong to the linear space $H(U)$ of all analytic functions on U take the form

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \ \ and \ \ g(z) = \sum_{n=1}^{\infty} b_n z^n.$$

Thus for each $f \in S_H$ takes the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}, \quad z \in U. \tag{1}$$

Clunie and Sheil-Small proved that S_H is not compact and the necessary and sufficient condition for f to be locally univalent and sense-preserving in any simply connected domain $\triangle$ is that $|h'(z)| > |g'(z)|$ (see [1]).

In [10-11], a generalized differential operator was introduced as follows: $\mathcal{D}^k_{\lambda,\delta} f(z)$, where

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \ \ (z \in U)$$

as follows :

$$\mathcal{D}^k_{\lambda,\delta} f(z) = z + \sum_{n=2}^{\infty} [1 + (n-1)\lambda]^k C(\delta, n) a_n z^n, \ \ k \in \mathbb{N}_0, \lambda \geq 0, \ \delta \geq 0, \tag{2}$$

where

$$C(\delta, n) = \binom{n+\delta-1}{\delta} = \frac{\Gamma(n+\delta)}{\Gamma(n)\Gamma(\delta+1)}.$$

This operator was later defined for second time by the authors in [2], without noticing that this operator has been given earlier by Al-Shaqsi and Darus [10] and further studied in [11].

Remark 1.1. When $\lambda = 1, \delta = 0$ we get Sălăgean differential operator [3], $k = 0$ gives Ruscheweyh operator [4], $\delta = 0$ implies Al-Oboudi differential operator of order (k) [5] and when $\lambda = 1$ operator (2) reduces to Al-Shaqsi and Darus differential operator [6] .

In the following definitions, we introduce new classes of analytic functions containing the generalized differential operator (2):

Definition 1.1. *Let $f(z)$ of the form (1). Then $f(z) \in HS^k(\lambda, \delta, \mu)$ if and only if*

$$\sum_{n=2}^{\infty}(n-\mu)[1+(n-1)\lambda]^k C(\delta, n)[|a_n|+|b_n|] \leq (1-\mu)(1-|b_1|), \;\; 0 \leq \mu < 1, \;\; |b_1| < 1,$$

for all $z \in U$.

Definition 1.2. *Let $f(z)$ of the form (1). Then $f(z) \in HC^k(\lambda, \delta, \mu)$ if and only if*

$$\sum_{n=2}^{\infty} n(n-\mu)[1+(n-1)\lambda]^k C(\delta, n)[|a_n|+|b_n|] \leq (1-\mu)(1-|b_1|), \;\; 0 \leq \mu < 1, \;\; |b_1| < 1,$$

for all $z \in U$.

Remark 1.1. Note that

$$HS^0(\lambda, 0, \mu) \equiv HS(\mu), \quad \text{and} \;\; HC^0(\lambda, 0, \mu) \equiv HC(\mu)$$

where the subclasses $HS(\mu)$ and $HC(\mu)$ are studied in [7]. And

$$HS^0(\lambda, 0, 0) \equiv HS, \quad \text{and} \;\; HC^0(\lambda, 0, 0) \equiv HC$$

where the subclasses HS and HC introduced in [8].

We need the next definition as follows:

Definition 1.3. *Let $F(z) = H(z) + \overline{G(z)}$ where $H(z) = z + \sum_{n=2}^{\infty} A_n z^n$ and $G(z) = \sum_{n=1}^{\infty} B_n z^n$. Then the generalized $\rho-$neighborhood of f to be the set*

$$N_\rho^k(f) = \{F : \sum_{n=2}^{\infty}(n-\mu)[1+(n-1)\lambda]^k C(\delta, n)(|a_n-A_n|+|b_n-B_n|)+(1-\mu)|b_1-B_1|$$

$$\leq (1-\mu)\rho\}.$$

Note that when $k = 0, \mu = 0$ and $\delta = 0$, we receive the set

$$N_\rho^0(f) = \{F : \sum_{n=2}^{\infty} n(|a_n - A_n| + |b_n - B_n|) + (1 - \mu)|b_1 - B_1| \leq \rho\}$$

which defined in [9]. And when $k = 0$ and $\delta = 0$ we pose the set

$$N_\rho^0(f) = \{F : \sum_{n=2}^{\infty}(n-\mu)(|a_n - A_n| + |b_n - B_n|) + (1-\mu)|b_1 - B_1| \leq (1-\mu)\rho\}$$

which defined in [7].

2. Main Results

In this section, we establish some properties of the classes $HS^k(\lambda, \delta, \mu)$ and $HC^k(\lambda, \delta, \mu)$ by obtaining the coefficient bonds. The next results come from the Definitions 1.1 and 1.2.

Theorem 2.1. *Let* $0 \leq \mu_1 \leq \mu_2 < 1$. *Then*

$$(i) \quad HS^k(\lambda, \delta, \mu_2) \subset HS^k(\lambda, \delta, \mu_1),$$
$$(ii) \quad HC^k(\lambda, \delta, \mu_2) \subset HC^k(\lambda, \delta, \mu_1).$$

Theorem 2.2. *Let the inequality*

$$|a_n| + |b_n| \leq \frac{(1-\mu)(1-|b_1|)}{(n-\mu)[1+(n-1)\lambda]^k C(\delta, n)}, \quad 0 \leq \mu < 1, \quad |b_1| < 1, \quad (z \in U)$$

be satisfied. Then f *belongs to the class* $HS^k(\lambda, \delta, \mu)$. *The result is sharp.*

Theorem 2.3. *Let the inequality*

$$|a_n| + |b_n| \leq \frac{(1-\mu)(1-|b_1|)}{n(n-\mu)[1+(n-1)\lambda]^k C(\delta, n)}, \quad 0 \leq \mu < 1, \quad |b_1| < 1, \quad z \in U$$

be satisfied. Then f *belongs to the class* $HC^k(\lambda, \delta, \mu)$. *The result is sharp.*

Theorem 2.4. $HS^k(\lambda, \delta, \mu) \subset HS^k(\lambda, \delta, 0)$ *and* $HC^k(\lambda, \delta, \mu) \subset HC^k(\lambda, \delta, 0)$.

Proof. Since for $0 \leq \mu < 1$ we have

$$\sum_{n=2}^{\infty} n[1+(n-1)\lambda]^k C(\delta, n)[|a_n| + |b_n|] \leq$$

$$\sum_{n=2}^{\infty}\frac{(n-\mu)}{(1-\mu)}[1+(n-1)\lambda]^k C(\delta,n)[|a_n|+|b_n|]\le(1-|b_1|)$$

and

$$\sum_{n=2}^{\infty}n^2[1+(n-1)\lambda]^k C(\delta,n)[|a_n|+|b_n|]\le$$

$$\sum_{n=2}^{\infty}\frac{n(n-\mu)}{(1-\mu)}[1+(n-1)\lambda]^k C(\delta,n)[|a_n|+|b_n|]\le(1-|b_1|)$$

we obtain the proof of the theorem.

Next we discuss the following properties:

Theorem 2.5. *The class $HS^k(\lambda,\delta,\mu)$ consists of locally univalent sense preserving harmonic mappings.*

Proof. Let $f\in HS^k(\lambda,\delta,\mu)$. For $z_1,z_2\in U$ such that $z_1\ne z_2$ our aim is to prove that $|f(z_1)-f(z_2)|>0$.

$$\begin{aligned}\Big|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\Big| &\ge 1-\Big|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\Big|\\ &=1-\Big|\frac{\sum_{n=1}^{\infty}b_n(z_1^n-z_2^n)}{(z_1-z_2)+\sum_{n=2}^{\infty}a_n(z_1^n-z_2^n)}\Big|\\ &>1-\frac{\sum_{n=1}^{\infty}|b_n|}{1-\sum_{n=2}^{\infty}|a_n|}\\ &\ge 1-\frac{\sum_{n=1}^{\infty}(n-\mu)[1+(n-1)\lambda]^k C(\delta,n)\frac{|b_n|)}{(1-\mu)(1-|b_1|}}{1-\sum_{n=2}^{\infty}(n-\mu)[1+(n-1)\lambda]^k C(\delta,n)\frac{|a_n|}{(1-\mu)(1-|b_1|)}}\\ &>0,\end{aligned}$$

where $a_1=1$. Hence f is univalent. Next we show that f is sense preserving mapping.

$$\begin{aligned}|h'(z)|-|g'(z)| &\ge 1-\sum_{n=2}^{\infty}n|a_n||z|^{n-1}-|b_1|-\sum_{n=2}^{\infty}n|b_n||z|^{n-1}\\ &>(1-|b_1|)-[\sum_{n=2}^{\infty}n(|a_n|+|b_n|)]\\ &>(1-|b_1|)-(1-|b_1|)\\ &=0.\end{aligned}$$

Hence $|h'(z)| - |g'(z)| > 0$. This completes the proof.

In the same way we obtain the following result.

Corollary 2.6. *The class $HC^k(\lambda, \delta, \mu)$ consists of locally univalent sense preserving harmonic mappings.*

Theorem 2.7. *Let $f \in HS^k(\lambda, \delta, \mu)$. Then*

$$|\mathcal{D}^k_{\lambda,\delta} f(z)| \leq (1 + \frac{|b_1|}{\delta})|z| + \frac{(1-\mu)(1-|b_1|)}{(2-\mu)}|z|^2$$

and

$$|\mathcal{D}^k_{\lambda,\delta} f(z)| \geq (1 - \frac{|b_1|}{\delta})|z| - \frac{(1-\mu)(1-|b_1|)}{(2-\mu)}|z|^2.$$

Proof. Let $f \in HS^k(\lambda, \delta, \mu)$ then we have

$$(2-\mu)\sum_{n=2}^{\infty}[1+(n-1)\lambda]^k C(\delta,n)[|a_n| + |b_n|] \leq$$

$$\sum_{n=2}^{\infty}(n-\mu)[1+(n-1)\lambda]^k C(\delta,n)[|a_n| + |b_n|] \leq (1-\mu)(1-|b_1|)$$

implies that

$$\sum_{n=2}^{\infty}[1+(n-1)\lambda]^k C(\delta,n)[|a_n| + |b_n|] \leq \frac{(1-\mu)(1-|b_1|)}{(2-\mu)}.$$

Applying this inequality in the following assertion, we obtain

$$\begin{aligned}
|\mathcal{D}^k_{\lambda,\delta} f(z)| &= |z + \sum_{n=2}^{\infty}[1+(n-1)\lambda]^k C(\delta,n) a_n z^n + \sum_{n=1}^{\infty}[1+(n-1)\lambda]^k C(\delta,n) b_n \overline{z}^n| \\
&\leq (1 + \frac{|b_1|}{\delta})|z| + \sum_{n=2}^{\infty}[1+(n-1)\lambda]^k C(\delta,n)(|a_n| + |b_n|)|z|^n \\
&\leq (1 + \frac{|b_1|}{\delta})|z| + \sum_{n=2}^{\infty}[1+(n-1)\lambda]^k C(\delta,n)(|a_n| + |b_n|)|z|^2 \\
&\leq (1 + \frac{|b_1|}{\delta})|z| + \frac{(1-\mu)(1-|b_1|)}{(2-\mu)}|z|^2.
\end{aligned}$$

Also, on the other hand we obtain

$$\begin{aligned}|\mathcal{D}^k_{\lambda,\delta}f(z)| &\geq (1-\frac{|b_1|}{\delta})|z| - \sum_{n=2}^{\infty}[1+(n-1)\lambda]^k C(\delta,n)(|a_n|+|b_n|)|z|^n \\ &\geq (1-\frac{|b_1|}{\delta})|z| - \sum_{n=2}^{\infty}[1+(n-1)\lambda]^k C(\delta,n)(|a_n|+|b_n|)|z|^2 \\ &\geq (1-\frac{|b_1|}{\delta})|z| - \frac{(1-\mu)(1-|b_1|)}{(2-\mu)}|z|^2.\end{aligned}$$

In similar manner we can prove the following result.

Theorem 2.8. *Let* $f \in HC^k(\lambda,\delta,\mu)$. *Then*

$$|\mathcal{D}^k_{\lambda,\delta}f(z)| \leq (1+\frac{|b_1|}{\delta})|z| + \frac{(1-\mu)(1-|b_1|)}{2(2-\mu)}|z|^2$$

and

$$|\mathcal{D}^k_{\lambda,\delta}f(z)| \geq (1-\frac{|b_1|}{\delta})|z| - \frac{(1-\mu)(1-|b_1|)}{2(2-\mu)}|z|^2.$$

Theorem 2.9. *Let* f *of the form (1) belongs to* $HC^k(\lambda,\delta,\mu)$. *If* $\rho \leq 1$ *then* $N^k_\rho(f) \subset HS^k(\lambda,\delta,\mu)$.

Proof. Let

$$f(z) = z + \sum_{n=2}^{\infty}[a_n z^n + \overline{b_n z^n}] + \overline{b_1 z}$$

and

$$F(z) = z + \sum_{n=2}^{\infty}[A_n z^n + \overline{B_n z^n}] + \overline{B_1 z}.$$

Let $f \in HC_k(\mu)$ and $F \in N^k_\rho(f)$ this give

$$(1-\mu)\rho \geq \sum_{n=2}^{\infty}(n-\mu)[1+(n-1)\lambda]^k C(\delta,n)(|a_n-A_n|+|b_n-B_n|)+(1-\mu)|b_1-B_1|$$

$$= \sum_{n=2}^{\infty}(n-\mu)[1+(n-1)\lambda]^k C(\delta,n)(|A_n-a_n|+|B_n-b_n|)+(1-\mu)|B_1-b_1|$$

$$\geq \sum_{n=2}^{\infty}(n-\mu)[1+(n-1)\lambda]^k C(\delta,n)(|A_n|-|a_n|+|B_n|-|b_n|)+(1-\mu)(|B_1|-|b_1|)$$

$$\geq \sum_{n=2}^{\infty}(n-\mu)[1+(n-1)\lambda]^k C(\delta,n)(|A_n|+|B_n|)+(1-\mu)|B_1|$$

$$-\Big(\sum_{n=2}^{\infty}(n-\mu)[1+(n-1)\lambda]^k C(\delta,n)(|a_n|+|b_n|)+(1-\mu)|b_1|\Big)$$

$$\geq \sum_{n=2}^{\infty}(n-\mu)[1+(n-1)\lambda]^k C(\delta,n)(|A_n|+|B_n|)+(1-\mu)|B_1|$$

$$-\Big(\sum_{n=2}^{\infty}n(n-\mu)[1+(n-1)\lambda]^k C(\delta,n)(|a_n|+|b_n|)+(1-\mu)|b_1|\Big)$$

$$\geq \sum_{n=2}^{\infty}(n-\mu)[1+(n-1)\lambda]^k C(\delta,n)(|A_n|+|B_n|)+(1-\mu)|B_1|-(1-\mu).$$

Thus we obtain that

$$\sum_{n=2}^{\infty}(n-\mu)[1+(n-1)\lambda]^k C(\delta,n)(|A_n|+|B_n|)\leq(1-\mu)(1-|B_1|),$$

when $\rho \leq 1$. Hence $F \in HS_k(\mu)$.

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