SOME EXTENSIONS OF SUFFICIENT CONDITIONS FOR UNIVALENCE OF AN INTEGRAL OPERATOR ON THE CLASSES $\mathcal{T}_j, \mathcal{T}_{j,\mu}$ and $\mathcal{S}_j(p)$

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Abstract. In this paper, we consider the subclasses $\mathcal{T}_j, \mathcal{T}_{j,\mu}$ and $\mathcal{S}_j(p)$ $(j = 2, 3, \ldots)$, and generalize univalence conditions for integral operator $F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}$ of the analytic function f belonging to the classes $\mathcal{T}_2, \mathcal{T}_{2,\mu}$ and $\mathcal{S}(p)$.

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1. Introduction

Let $\mathcal{A}$ be the class of all functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

which are analytic in the open unit disk

$$\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}.$$

Consider

$$\mathcal{S} = \{f \in \mathcal{A} : f are univalent functions in \mathbb{U}\}.$$

Let $\mathcal{A}_j$ be the subclass of $\mathcal{A}$ consisting of functions f given by

$$f(z) = z + \sum_{k=j+1}^{\infty} a_k z^k \qquad (j \in \mathbb{N}_1^* := \mathbb{N} \setminus \{0, 1\} = \{2, 3, \ldots\}). \tag{1}$$

Let $\mathcal{T}$ be the univalent subclass of $\mathcal{A}$ consisting of functions f which satisfy

$$\left|\frac{z^2 f'(z)}{(f(z))^2} - 1\right| < 1 \qquad (z \in \mathbb{U}).$$

Let $\mathcal{T}_j$ be the subclass of $\mathcal{T}$ for which $f^{(k)}(0) = 0$ $(k = 2, 3, \ldots, j)$. Let $\mathcal{T}_{j,\mu}$ be the subclass of $\mathcal{T}_j$ consisting of functions f of the form (1) which satisfy

$$\left|\frac{z^2 f'(z)}{(f(z))^2} - 1\right| < \mu \qquad (z \in \mathbb{U}) \tag{2}$$

for some μ $(0 < \mu \leq 1)$, and let us denote $\mathcal{T}_{j,1} \equiv \mathcal{T}_j$.

For some real p with $0 < p \leq 2$, we define the subclass $\mathcal{S}(p)$ of $\mathcal{A}$ consisting of all functions f which satisfy

$$\left|\left(\frac{z}{f(z)}\right)''\right| \leq p \qquad (z \in \mathbb{U}). \tag{3}$$

In [7], Singh has shown that if $f \in \mathcal{S}(p)$, then f satisfies

$$\left|\frac{z^2 f'(z)}{(f(z))^2} - 1\right| \leq p\,|z|^2 \qquad (z \in \mathbb{U}). \tag{4}$$

Let $\mathcal{S}_j(p)$ be the subclass of $\mathcal{A}$ consisting of functions $f \in \mathcal{A}_j$ which satisfy (3) and

$$\left|\frac{z^2 f'(z)}{(f(z))^2} - 1\right| \leq p\,|z|^j \qquad (z \in \mathbb{U},\ j \in \mathbb{N}_1^*), \tag{5}$$

and let us denote by $\mathcal{S}_2(p) \equiv \mathcal{S}(p)$.

The subclasses $\mathcal{T}_j, \mathcal{T}_{j,\mu}$ and $\mathcal{S}_j(p)$ are introduced by Seenivasagan [5].

The following results will be required in our investigation.

General Schwarz Lemma. [3] *Let the function f be regular in the disk $\mathbb{U}_R = \{z \in \mathbb{C} : |z| < R\}$, with $|f(z)| < M$ for fixed M. If f has one zero with multiplicity order bigger than m for $z = 0$, then*

$$|f(z)| \leq \frac{M}{R^m}\,|z|^m \quad (z \in \mathbb{U}_R).$$

The equality can hold only if $f(z) = e^{i\theta}(M/R^m)\,z^m$, where θ is constant.

Theorem A. [4] *Let $\alpha \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0$ and $f \in \mathcal{A}$. If f satisfies*

$$\frac{1 - |z|^{2\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1 \qquad (z \in \mathbb{U}),$$

then, for any complex number β *with* $\mathrm{Re}(\beta) \geq \mathrm{Re}(\alpha)$, *the integral operator*

$$F_\beta(z) = \left\{ \beta \int_0^z t^{\beta-1} f'(t) dt \right\}^{\frac{1}{\beta}} \tag{6}$$

is in the class $\mathcal{S}$.

Theorem B. [1] *Let* $f_i \in \mathcal{T}_2$ *and*

$$f_i(z) = z + \sum_{k=3}^{\infty} a_k^i z^k \tag{7}$$

for $\forall\, i = \overline{1, n},\ n \in \mathbb{N}^*$. *If* $|f_i(z)| \leq 1$ $(z \in \mathbb{U})$, *then*

$$F_{\alpha,\beta}(z) = \left\{ \beta \int_0^z t^{\beta-1} \prod_{i=1}^{n} \left(\frac{f_i(t)}{t} \right)^{\frac{1}{\alpha}} dt \right\}^{\frac{1}{\beta}} \in \mathcal{S}, \tag{8}$$

where $\alpha, \beta \in \mathbb{C}$, $\mathrm{Re}(\alpha) \geq \frac{3n}{|\alpha|}$, *and* $\mathrm{Re}(\beta) \geq \mathrm{Re}(\alpha)$.

Theorem C. [1] *Let* f_i *defined by* (7) *be in the class* $\mathcal{T}_{2,\mu}$ *for* $\forall\, i = \overline{1, n},\ n \in \mathbb{N}^* := \mathbb{N} \setminus \{0\}$. *If*

$$|f_i(z)| \leq 1 \ (z \in \mathbb{U}),$$

then, for $\beta \in \mathbb{C}$, *the integral operator* $F_{\alpha,\beta}$ *defined by* (8) *is in the class* $\mathcal{S}$, *where* $\alpha, \beta \in \mathbb{C}$,

$$\mathrm{Re}(\alpha) \geq \frac{(\mu + 2)n}{|\alpha|},$$

and $\mathrm{Re}(\beta) \geq \mathrm{Re}(\alpha)$.

Theorem D. [1] *Let* f_i *defined by* (7) *be in the class* $\mathcal{S}(p)$ *for* $\forall\, i = \overline{1, n},\ n \in \mathbb{N}^*$. *If*

$$|f_i(z)| \leq 1 \ (z \in \mathbb{U}),$$

then, for $\beta \in \mathbb{C}$, *the integral operator* $F_{\alpha,\beta}$ *defined by* (8) *is in the class* $\mathcal{S}$, *where* $\alpha, \beta \in \mathbb{C}$,

$$\mathrm{Re}(\alpha) \geq \frac{(p + 2)n}{|\alpha|},$$

and $\mathrm{Re}(\beta) \geq \mathrm{Re}(\alpha)$.

Breaz and Owa [2] gave the extensions of Theorems B, C and D as follows.

Theorem B′. [2] *Let* f_i *defined by* (7) *be in the class* $\mathcal{T}_2$ *for* $\forall\, i = \overline{1,n}$, $n \in \mathbb{N}^*$. *If*

$$|f_i(z)| \leq M \ (M \geq 1\,;\ z \in \mathbb{U}),$$

then, for $\beta \in \mathbb{C}$, *the integral operator* $F_{\alpha,\beta}$ *defined by* (8) *is in the class* $\mathcal{S}$, *where* $\alpha, \beta \in \mathbb{C}$,

$$\mathrm{Re}(\alpha) \geq \frac{(2M+1)\,n}{|\alpha|},$$

and $\mathrm{Re}(\beta) \geq \mathrm{Re}(\alpha)$.

Theorem C′. [2] *Let* f_i *defined by* (7) *be in the class* $\mathcal{T}_{2,\mu}$ *for* $\forall\, i = \overline{1,n}$, $n \in \mathbb{N}^*$. *If*

$$|f_i(z)| \leq M \ (M \geq 1\,;\ z \in \mathbb{U}),$$

then, for $\beta \in \mathbb{C}$, *the integral operator* $F_{\alpha,\beta}$ *defined by* (8) *is in the class* $\mathcal{S}$, *where* $\alpha, \beta \in \mathbb{C}$,

$$\mathrm{Re}(\alpha) \geq \frac{((\mu+1)M+1)\,n}{|\alpha|},$$

and $\mathrm{Re}(\beta) \geq \mathrm{Re}(\alpha)$.

Theorem D′. [2] *Let* f_i *defined by* (7) *be in the class* $\mathcal{S}(p)$ *for* $\forall\, i = \overline{1,n}$, $n \in \mathbb{N}^*$. *If*

$$|f_i(z)| \leq M \ (M \geq 1\,;\ z \in \mathbb{U}),$$

then, for $\beta \in \mathbb{C}$, *the integral operator* $F_{\alpha,\beta}$ *defined by* (8) *is in the class* $\mathcal{S}$, *where* $\alpha, \beta \in \mathbb{C}$,

$$\mathrm{Re}(\alpha) \geq \frac{((p+1)M+1)\,n}{|\alpha|},$$

and $\mathrm{Re}(\beta) \geq \mathrm{Re}(\alpha)$.

In [6], Seenivasagan and Breaz considered the integral operator

$$F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}(z) = \left\{ \beta \int_0^z t^{\beta-1} \prod_{i=1}^{n} \left(\frac{f_i(t)}{t} \right)^{\frac{1}{\alpha_i}} dt \right\}^{\frac{1}{\beta}} \tag{9}$$

for $f_i \in \mathcal{A}_2$ $(i = 1, 2, \ldots, n)$ and $\alpha_1, \alpha_2, \ldots, \alpha_n, \beta \in \mathbb{C}$.

For $\alpha_1 = \alpha_2 = \ldots = \alpha_n = \alpha$, $F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}$ becomes the integral operator $F_{\alpha,\beta}$ defined in (8).

Seenivasagan and Breaz [6] gave the extensions Theorems C′ and D′ as follows.

Theorem C$''$. [6] *Let* $M \geq 1$, $f_i \in \mathcal{T}_{2,\mu_i}$ *defined by* (7), $\alpha_i, \beta \in \mathbb{C}$, $\mathrm{Re}(\beta) \geq \gamma$ *and*

$$\gamma := \sum_{i=1}^{n} \frac{(1+\mu_i)M+1}{|\alpha_i|} \quad (0 < \mu_i \leq 1, \textit{for all}\, i = 1,2,\ldots,n,\, n \in \mathbb{N}^*).$$

If

$$|f_i(z)| \leq M \quad (z \in \mathbb{U}, \quad i = 1,2,\ldots,n),$$

then the integral operator $F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}$ *defined by* (9) *is in* $\mathcal{S}$.

Theorem D$''$. [6] *Let* $M \geq 1$, $f_i \in \mathcal{S}(p)$ *defined by* (7), $\alpha_i, \beta \in \mathbb{C}$, $\mathrm{Re}(\beta) \geq \gamma_1$ *and*

$$\gamma_1 := \sum_{i=1}^{n} \frac{(1+p)M+1}{|\alpha_i|} \quad (\textit{for alli} = 1,2,\ldots,n,\, n \in \mathbb{N}^*).$$

If

$$|f_i(z)| \leq M \quad (z \in \mathbb{U}, \quad i = 1,2,\ldots,n),$$

then the integral operator $F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}$ *defined by* (9) *is in* $\mathcal{S}$.

In this paper, we obtain some generalizations of the results given by [1], [2], [5] and [6].

2. Main results

Theorem 1. *Let* f_i *defined by*

$$f_i(z) = z + \sum_{k=j+1}^{\infty} a_k^i z^k \tag{10}$$

be in the class $\mathcal{T}_j$ *for* $\forall\, i = \overline{1,n}$, $n \in \mathbb{N}^*$, $j \in \mathbb{N}_1^*$. *If*

$$|f_i(z)| \leq M_i \ (M_i \geq 1\,;\, z \in \mathbb{U}),$$

then the integral operator $F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}$ *defined by* (9) *is in the class* $\mathcal{S}$, *where* $\alpha, \beta \in \mathbb{C}$,

$$\mathrm{Re}(\alpha) \geq \sum_{i=1}^{n} \frac{2M_i+1}{|\alpha_i|},$$

and $\mathrm{Re}(\beta) \geq \mathrm{Re}(\alpha)$.

Proof. Define a function

$$h(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{f_i(t)}{t}\right)^{\frac{1}{\alpha_i}} dt.$$

Then we obtain

$$h'(z) = \prod_{i=1}^{n} \left(\frac{f_i(z)}{z}\right)^{\frac{1}{\alpha_i}}.$$

It is clear that $h(0) = h'(0) - 1 = 0$. Also, a simple computation yields

$$\frac{zh''(z)}{h'(z)} = \sum_{i=1}^{n} \frac{1}{\alpha_i} \left(\frac{zf_i'(z)}{f_i(z)} - 1\right). \tag{11}$$

From (11), we get

$$\frac{1-|z|^{2\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)} \left|\frac{zh''(z)}{h'(z)}\right| \leq \frac{1-|z|^{2\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)} \sum_{i=1}^{n} \frac{1}{|\alpha_i|} \left(\left|\frac{z^2 f_i'(z)}{(f_i(z))^2}\right| \left|\frac{f_i(z)}{z}\right| + 1\right). \tag{12}$$

From the hypothesis, we have $|f_i(z)| \leq M_i$ ($i = \overline{1,n}$; $z \in \mathbb{U}$), then by the general Schwarz lemma, we obtain that

$$|f_i(z)| \leq M_i\,|z| \quad (i = \overline{1,n}\,;\, z \in \mathbb{U}).$$

We apply this result in inequality (12), then we find

$$\begin{aligned}
\frac{1-|z|^{2\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)} \left|\frac{zh''(z)}{h'(z)}\right| &\leq \frac{1-|z|^{2\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)} \sum_{i=1}^{n} \frac{1}{|\alpha_i|} \left(\left|\frac{z^2 f_i'(z)}{(f_i(z))^2}\right| M_i + 1\right) \\
&\leq \frac{1-|z|^{2\mathrm{Re}(\alpha)}}{\mathrm{Re}(\alpha)} \sum_{i=1}^{n} \frac{1}{|\alpha_i|} \left(\left|\frac{z^2 f_i'(z)}{(f_i(z))^2} - 1\right| M_i + M_i + 1\right) \\
&\leq \frac{1}{\mathrm{Re}(\alpha)} \sum_{i=1}^{n} \frac{2M_i + 1}{|\alpha_i|} \\
&\leq 1
\end{aligned}$$

since $\mathrm{Re}(\alpha) \geq \sum_{i=1}^{n} \frac{2M_i+1}{|\alpha_i|}$. Applying Theorem A, we obtain that $F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}$ is univalent.

Corollary 2. *Let f_i defined by* (10) *be in the class $\mathcal{T}_j$ for $\forall\, i = \overline{1,n}$, $n \in \mathbb{N}^*$, $j \in \mathbb{N}_1^*$. If*

$$|f_i(z)| \leq M \ (M \geq 1\,;\, z \in \mathbb{U}),$$

then the integral operator $F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}$ *defined by* (9) *is in the class* $\mathcal{S}$, *where* $\alpha, \beta \in \mathbb{C}$,

$$\operatorname{Re}(\alpha) \geq \sum_{i=1}^{n} \frac{2M+1}{|\alpha_i|},$$

and $\operatorname{Re}(\beta) \geq \operatorname{Re}(\alpha)$.

Proof. In Theorem 1, we consider $M_1 = M_2 = \cdots = M_n = M$.

Corollary 3. *Let* f_i *defined by* (10) *be in the class* $\mathcal{T}_j$ *for* $\forall\, i = \overline{1,n}$, $n \in \mathbb{N}^*$, $j \in \mathbb{N}_1^*$. *If*

$$|f_i(z)| \leq M \ (M \geq 1\,;\, z \in \mathbb{U}),$$

then the integral operator $F_{\alpha,\beta}$ *defined by* (8) *is in the class* $\mathcal{S}$, *where* $\alpha, \beta \in \mathbb{C}$,

$$\operatorname{Re}(\alpha) \geq \frac{(2M+1)\,n}{|\alpha|},$$

and $\operatorname{Re}(\beta) \geq \operatorname{Re}(\alpha)$.

Proof. In Corollary 2, we consider $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha$.

Corollary 4. *If we set* $j = 2$ *in Corollary* 3, *then we have Theorem* B$'$.

Corollary 5. *If we set* $j = 2$ *and* $M = 1$ *in Corollary* 3, *then we have Theorem* B.

The proofs of Theorems 6 and 13 below are much akin to that of Theorem 1, which we have detailed above fairly fully.

Theorem 6. *Let* f_i *defined by* (10) *be in the class* $\mathcal{T}_{j,\mu_i}$ *for* $\forall\, i = \overline{1,n}$, $n \in \mathbb{N}^*$, $j \in \mathbb{N}_1^*$. *If*

$$|f_i(z)| \leq M_i \ (M_i \geq 1\,;\, z \in \mathbb{U}),$$

then the integral operator $F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}$ *defined by* (9) *is in the class* $\mathcal{S}$, *where* $\alpha, \beta \in \mathbb{C}$,

$$\operatorname{Re}(\alpha) \geq \sum_{i=1}^{n} \frac{(\mu_i+1)M_i+1}{|\alpha_i|},$$

and $\operatorname{Re}(\beta) \geq \operatorname{Re}(\alpha)$.

Corollary 7. *Let* f_i *defined by* (10) *be in the class* $\mathcal{T}_{j,\mu_i}$ *for* $\forall\, i = \overline{1,n}$, $n \in \mathbb{N}^*$, $j \in \mathbb{N}_1^*$. *If*

$$|f_i(z)| \leq M \ (M \geq 1\,;\, z \in \mathbb{U}),$$

then the integral operator $F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}$ *defined by* (9) *is in the class* $\mathcal{S}$, *where* $\alpha, \beta \in \mathbb{C}$,

$$\mathrm{Re}(\alpha) \geq \sum_{i=1}^{n} \frac{(\mu_i + 1)M + 1}{|\alpha_i|},$$

and $\mathrm{Re}(\beta) \geq \mathrm{Re}(\alpha)$.

Proof. In Theorem 6, we consider $M_1 = M_2 = \cdots = M_n = M$.

Corollary 8. *Let* f_i *defined by* (10) *be in the class* $\mathcal{T}_{j,\mu_i}$ *for* $\forall\, i = \overline{1,n}$, $n \in \mathbb{N}^*$, $j \in \mathbb{N}_1^*$. *If*

$$|f_i(z)| \leq M \ (M \geq 1\, ;\, z \in \mathbb{U}),$$

then the integral operator $F_{\alpha,\beta}$ *defined by* (8) *is in the class* $\mathcal{S}$, *where* $\alpha, \beta \in \mathbb{C}$,

$$\mathrm{Re}(\alpha) \geq \sum_{i=1}^{n} \frac{(\mu_i + 1)M + 1}{|\alpha|},$$

and $\mathrm{Re}(\beta) \geq \mathrm{Re}(\alpha)$.

Proof. In Corollary 7, we consider $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha$. This generalizes Theorem 1 in [5].

Corollary 9. *Let* f_i *defined by* (10) *be in the class* $\mathcal{T}_{j,\mu}$ *for* $\forall\, i = \overline{1,n}$, $n \in \mathbb{N}^*$, $j \in \mathbb{N}_1^*$. *If*

$$|f_i(z)| \leq M \ (M \geq 1\, ;\, z \in \mathbb{U}),$$

then the integral operator $F_{\alpha,\beta}$ *defined by* (8) *is in the class* $\mathcal{S}$, *where* $\alpha, \beta \in \mathbb{C}$,

$$\mathrm{Re}(\alpha) \geq \frac{((\mu + 1)M + 1)\, n}{|\alpha|},$$

and $\mathrm{Re}(\beta) \geq \mathrm{Re}(\alpha)$.

Proof. In Corollary 8, we consider $\mu_1 = \mu_2 = \cdots = \mu_n = \mu$.

Corollary 10. *If we set* $j = 2$ *in Corollary* 7, *then we have Theorem* C$''$.

Corollary 11. *If we set* $j = 2$ *in Corollary* 9, *then we have Theorem* C$'$.

Corollary 12. *If we set* $j = 2$ *and* $M = 1$ *in Corollary* 9, *then we have Theorem* C.

Theorem 13. *Let* f_i *defined by* (10) *be in the class* $\mathcal{S}_j(p_i)$ *for* $\forall\, i = \overline{1,n}$, $n \in \mathbb{N}^*$, $j \in \mathbb{N}_1^*$. *If*

$$|f_i(z)| \leq M_i \ (M_i \geq 1\,;\, z \in \mathbb{U}),$$

then the integral operator $F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}$ *defined by* (9) *is in the class* $\mathcal{S}$, *where* $\alpha, \beta \in \mathbb{C}$,

$$\mathrm{Re}(\alpha) \geq \sum_{i=1}^{n} \frac{(p_i+1)M_i+1}{|\alpha_i|},$$

and $\mathrm{Re}(\beta) \geq \mathrm{Re}(\alpha)$.

Corollary 14. *Let* f_i *defined by* (10) *be in the class* $\mathcal{S}_j(p_i)$ *for* $\forall\, i = \overline{1,n}$, $n \in \mathbb{N}^*$, $j \in \mathbb{N}_1^*$. *If*

$$|f_i(z)| \leq M \ (M \geq 1\,;\, z \in \mathbb{U}),$$

then the integral operator $F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}$ *defined by* (9) *is in the class* $\mathcal{S}$, *where* $\alpha, \beta \in \mathbb{C}$,

$$\mathrm{Re}(\alpha) \geq \sum_{i=1}^{n} \frac{(p_i+1)M+1}{|\alpha_i|},$$

and $\mathrm{Re}(\beta) \geq \mathrm{Re}(\alpha)$.

Proof. In Theorem 13, we consider $M_1 = M_2 = \cdots = M_n = M$.

Corollary 15. *Let* f_i *defined by* (10) *be in the class* $\mathcal{S}_j(p_i)$ *for* $\forall\, i = \overline{1,n}$, $n \in \mathbb{N}^*$, $j \in \mathbb{N}_1^*$. *If*

$$|f_i(z)| \leq M \ (M \geq 1\,;\, z \in \mathbb{U}),$$

then the integral operator $F_{\alpha,\beta}$ *defined by* (8) *is in the class* $\mathcal{S}$, *where* $\alpha, \beta \in \mathbb{C}$,

$$\mathrm{Re}(\alpha) \geq \sum_{i=1}^{n} \frac{(p_i+1)M+1}{|\alpha|},$$

and $\mathrm{Re}(\beta) \geq \mathrm{Re}(\alpha)$.

Proof. In Corollary 14, we consider $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha$.

Corollary 16. *Let* f_i *defined by* (10) *be in the class* $\mathcal{S}_j(p)$ *for* $\forall\, i = \overline{1,n}$, $n \in \mathbb{N}^*$, $j \in \mathbb{N}_1^*$. *If*

$$|f_i(z)| \leq M \ (M \geq 1\,;\, z \in \mathbb{U}),$$

then the integral operator $F_{\alpha_1,\alpha_2,\ldots,\alpha_n,\beta}$ *defined by* (9) *is in the class* $\mathcal{S}$, *where* $\alpha, \beta \in \mathbb{C}$,

$$\mathrm{Re}(\alpha) \geq \sum_{i=1}^{n} \frac{(p+1)M+1}{|\alpha_i|},$$

and $\mathrm{Re}(\beta) \geq \mathrm{Re}(\alpha)$.

Proof. In Corollary 14, we consider $p_1 = p_2 = \cdots = p_n = p$.

Corollary 17. *Let* f_i *defined by* (10) *be in the class* $\mathcal{S}_j(p)$ *for* $\forall\, i = \overline{1,n}$, $n \in \mathbb{N}^*$, $j \in \mathbb{N}_1^*$. *If*

$$|f_i(z)| \leq M \ (M \geq 1\,;\, z \in \mathbb{U}),$$

then the integral operator $F_{\alpha,\beta}$ *defined by* (8) *is in the class* $\mathcal{S}$, *where* $\alpha, \beta \in \mathbb{C}$,

$$\mathrm{Re}(\alpha) \geq \frac{((p+1)M+1)\,n}{|\alpha|},$$

and $\mathrm{Re}(\beta) \geq \mathrm{Re}(\alpha)$.

Proof. If we consider $p_1 = p_2 = \cdots = p_n = p$ in Corollary 15 or $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha$ in Corollary 16, then we get desired result. This generalizes Theorem 2 in [5].

Corollary 18. *If we set* $j = 2$ *in Corollary* 16, *then we have Theorem* D$''$.

Corollary 19. *If we set* $j = 2$ *in Corollary* 17, *then we have Theorem* D$'$.

Corollary 20. *If we set* $j = 2$ *and* $M = 1$ *in Corollary* 17, *then we have Theorem* D.

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